The Humongous Book of Basic Math & Pre-Algebra Problems

Over 800 basic math and pre-algebra problems with comprehensive solutions for all major topics.

Translated for People who don't Speak Math!!

by W. Michael Kelley

The Humongous Book of Basic Math & Pre-Algebra Problems

Translated for People who don't Speak Math!!

by W. Michael Kelley

ALPHA
A member of Penguin Group (USA) Inc.

ALPHA BOOKS

Published by the Penguin Group

Penguin Group (USA) Inc., 375 Hudson Street, New York, New York 10014, USA

Penguin Group (Canada), 90 Eglinton Avenue East, Suite 700, Toronto, Ontario M4P 2Y3, Canada (a division of Pearson Penguin Canada Inc.)

Penguin Books Ltd., 80 Strand, London WC2R 0RL, England

Penguin Ireland, 25 St. Stephen's Green, Dublin 2, Ireland (a division of Penguin Books Ltd.)

Penguin Group (Australia), 250 Camberwell Road, Camberwell, Victoria 3124, Australia (a division of Pearson Australia Group Pty. Ltd.)

Penguin Books India Pvt. Ltd., 11 Community Centre, Panchsheel Park, New Delhi—110 017, India

Penguin Group (NZ), 67 Apollo Drive, Rosedale, North Shore, Auckland 1311, New Zealand (a division of Pearson New Zealand Ltd.)

Penguin Books (South Africa) (Pty.) Ltd., 24 Sturdee Avenue, Rosebank, Johannesburg 2196, South Africa

Penguin Books Ltd., Registered Offices: 80 Strand, London WC2R 0RL, England

International Standard Book Number: 978-1-61564-083-6
Library of Congress Catalog Card Number: 2011902716

13 8 7 6 5 4 3

Interpretation of the printing code: The rightmost number of the first series of numbers is the year of the book's printing; the rightmost number of the second series of numbers is the number of the book's printing. For example, a printing code of 11-1 shows that the first printing occurred in 2011.

Printed in the United States of America

Contents

Introduction

Are you in a Basic Math or Pre-Algebra class? Are you trying to help a friend or loved one with a math class but discovering that you don't remember everything you used to know about math? Are you the curious sort who used to dislike math but now want to figure it out, once and for all?

If you answered YES to any of these questions, then you NEED this book. Here's why:

Fact #1: The best way to develop basic math and pre-algebra skills is by working out math problems.

There's no denying it. If you could figure this class out just by reading the textbook or taking good notes in class, everybody would pass with flying colors. Unfortunately, the harsh truth is that you have to buckle down and work problems out until your fingers are numb.

Fact #2: Most textbooks only tell you WHAT the answers to their practice problems are but not HOW to do them!

Sure your textbook may have 175 problems for every topic, but most of them only give you the answers. That means if you don't get the answer right you're totally out of luck! Knowing you're wrong is no help at all if you don't know WHY you're wrong.

Math textbooks sit on a huge throne, like the Great and Powerful Oz and say, "Nope, try again," and you do. Over and over, usually getting the problem wrong. What a delightful way to learn! (Let's not even get into why they only tell you the answers to the odd problems. Does that mean the book's actual AUTHOR didn't even feel like working out the even ones?)

Fact #3: Even when math books try to show you the steps for a problem, they do a lousy job.

Math people love to skip steps. You'll be following along fine with an explanation and then all of a sudden BAM, you're lost. You'll think to yourself, "How did they do that?" or "Where the heck did that 42 come from? It wasn't there in the last step!"

Why do math textbooks insist that in order to work out a problem on page 200, you'd better know pages 1 through 199 like the back of your hand? You don't want to spend the rest of your life on homework! You just want to know what you're supposed to do when you're dividing by a number that has a decimal point in it, which you'll learn in Chapter 10.

Fact #4: Reading lists of facts is fun for a while, but then it gets old. Enough with this list—let's cut to the chase.

All of my notes are off to the side like this and point to the parts of the book I'm trying to explain.

Just about every single kind of basic math or pre-algebra problem you could possibly run into is in here. After all, this book is HUMONGOUS! If 850 problems aren't enough, then you've got some kind of crazy math hunger, my friend, and I'd seek professional help.

This practice book was good at first, but to make it GREAT, I went through and worked out all the problems and took notes in the margins when I thought something was confusing or needed a little more explanation. I also drew little skulls next to the hardest problems, so you'd know not to freak out if they were too challenging. After all, if you're working on a problem and you're totally stumped, isn't it better to know that the problem is SUPPOSED to be hard? It's reassuring, at least for me.

I think you'll be pleasantly surprised by how detailed the answer explanations are, and I hope you'll find my little notes helpful along the way. Call me crazy, but I think that people who WANT to learn basic math and pre-algebra and are willing to spend the time drilling their way through practice problems should actually be able to figure the problems out and learn as they go, but that's just my 2¢.

Good luck and make sure to come visit my website at www.calculus-help.com. If you feel so inclined, drop me an e-mail and give me your 2¢. (Not literally, though—real pennies clog up the Internet pipes.)

—Mike Kelley

Acknowledgements

Special thanks to Pearson Publishing, who have supported the Humongous Books from the very beginning and continue to print massively large books for me despite the purported death of paper publications.

Thanks to Sue Strickland and Dave Herzog, who have tirelessly worked for nearly a year to proofread, edit, and check all of these problems as I developed them. I also want to thank the anonymous employee of the U.S. Postal Service who applied postage to one of Sue's packets of proofread chapters in a truly unique way. I have never seen a 9 × 12 envelope covered in 1¢ stamps before, and it was truly a thing of beauty.

Most of all, I want to thank my family, who has stood by me and supported me with each passing book. This marks my eleventh book in twelve years, so we're talking about a heap of support.

Trademarks

Dedication

For my wife and kids, the people who know me best but manage to love me anyway:

For Sara, who has the brightest smile the world has ever seen and spoke in complete sentences before she was three. My emotional little girl, who feels everything so deeply and is deathly afraid of very old ducks. (Don't ask.)

For Erin, the loveable goofball who is as tough as nails and loves her family more than anything. I can't bear to think of the day you stop announcing, "I want to sit by Dad!"

For Nick, the second grader with the crazy math brain. We found a green field hidden in the woods last week, and I will always remember what you said: "This will be our secret place! We can come here at night, lie on the grass, and watch the stars together!" Your heart is my heart.

Finally, for Lisa, who only loves me more the harder things get. 2010 is a year of bad memories, of getting laid off and scrounging for a torturous job. 2010 is also the best year of my life, because the darker things got, the brighter you shined. God's greatest gift to me is you, but no one said God didn't have a sense of humor. I snore like a freight train in a tornado. Sorry about that.

Chapter 1

NUMBERS

Easy as 1, 2, 3

As alphabetic systems use a fixed, finite series of symbols called letters to form words and represent language, numeric systems use symbols called digits to represent numbers. The Hindu-Arabic number system is, for all intents and purposes, the exclusive number system in the modern world. Using the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9, it is capable of easily representing any numeric value, regardless of how large or small. In this chapter, your study of basic mathematics and pre-algebra begins with an exploration of numbers mechanically and conceptually.

Ten simple digits can construct any number value. Think about that for a minute—pretty powerful stuff. However, with great power comes great responsibility (or so says a certain super hero who got bitten by a spider). All of those numbers have to go in very specific places, or the number system breaks down.

You'll start this chapter with the place values of numbers and then work into rounding numbers to one of those specific place values. Next, you'll look at the ways numbers are classified (even, odd, whole, real, etc.), and then finish up with common math symbols you'll find stuck between numbers for the rest of the book.

Place Values

Hundreds, thousands, millions, and billions

1.1 List the place values for each digit in the number 17 and use that information to identify the number.

The rightmost digit of a number is in the "ones place." Moving left, the next digit in the number is in the "tens place." The number 17 is comprised of two digits:

- 7 is in the ones place
- 1 is in the tens place

The number 17 is read "seventeen."

The ones digit is on the right only when there's no decimal in the number. In that case, the ones digit is the first digit left of the decimal. (More on decimals starting in Chapter 10.)

1.2 List the place values for each digit in the number 285 and use that information to identify the number.

As Problem 1.1 explains, the rightmost digit in a number represents the ones place, and the next digit as you move left represents the tens place. Continuing to move left, the next digit represents the hundreds place. The number 285 consists of three digits:

- 5 is in the ones place
- 8 is in the tens place
- 2 is in the hundreds place

The number 285 is read "two hundred eighty-five."

Not "two hundred AND eighty-five." Leave out "and" because that word represents a decimal point.

1.3 List the place values for each digit in the number 3,460 and use that information to identify the number.

Numbers larger than three digits usually include commas, as they make the place values of the digits easier to identify. Working left from the ones place, a comma is inserted after every group of three digits. In this problem, a comma separates the hundreds place (the third digit from the right) from the thousands place (the fourth digit from the right):

- 0 is in the ones place
- 6 is in the tens place
- 4 is in the hundreds place
- 3 is in the thousands place

The number 3,460 is read "three thousand, four hundred sixty." Notice that a comma separates the words "three thousand" from the words "four hundred," just as a comma separates the digit 3 from the digit 4.

1.4 List the place values for each digit in the number 62,009 and use that information to identify the number.

Two digits of the number 62,009 appear before the comma. As Problem 1.3 explains, the digit 2 is in the thousands place. To its left, the digit 6 appears in the ten thousands place. Notice that zeroes appear both in the hundreds and tens place.

- 9 is in the ones place
- 0 is in the tens place
- 0 is in the hundreds place
- 2 is in the thousands place
- 6 is in the ten thousands place

Without those zeroes the number would be 629.

The number 62,009 is read "sixty-two thousand, nine."

1.5 List the place values for each digit in the number 407,211 and use that information to identify the number.

Moving from right to left, the third digit of a number represents the hundreds place. Similarly, the digit three places further to the left represents the hundred thousands place.

- 1 is in the ones place
- 1 is in the tens place
- 2 is in the hundreds place
- 7 is in the thousands place
- 0 is in the ten thousands place
- 4 is in the hundred thousands place

The number 407,211 is read "four hundred seven thousand, two hundred eleven."

1.6 List the place values for each digit in the number 8,450,700 and use that information to identify the number.

Notice that this number contains two commas that separate the digits into groups of three, as you work from right to left. With the introduction of each new comma, a new term is used to describe its place values. For example, in Problem 1.3, the word "thousand" is introduced to describe the digit left of the comma. Here, the term "million" is used to describe the digit left of the second comma. One million is equal to one thousand thousands.

1,000 = one thousand

1,000,000 = one million

1,000,000,000 = one billion

1,000,000,000,000 = one trillion

- 0 is in the ones place
- 0 is in the tens place
- 7 is in the hundreds place
- 0 is in the thousands place

- 5 is in the ten thousands place
- 4 is in the hundred thousands place
- 8 is in the millions place

The number 8,450,700 is read "eight million, four hundred fifty thousand, seven hundred."

1.7 Write the value numerically: eight hundred thirteen.

Problems 1.1–1.6 demonstrate how to identify the place values of digits in a given number. In each case, you work from the rightmost digit, which represents the ones place. Similarly, given a number written as a word or phrase, you should begin with the rightmost word—in this case, "thirteen." The number thirteen (13) has a 3 in the ones place and a 1 in the tens place. Moving left in the phrase, the words "eight hundred" indicate an 8 should occupy the hundreds place.

Thus, the numerical representation of eight hundred thirteen is 813.

1.8 Write the value numerically: eighty-five thousand, six hundred fifty-three.

The rightmost word in this phrase is "fifty-three," which indicates a 3 in the ones place and a 5 in the tens place. Moving left, the words "six hundred" direct you to place a 6 in the hundreds place.

Continuing to move left in the phrase, you cross a comma, so a comma should be placed in the number at this location. Finally, the words "eighty-five thousand" prompt you to place a 5 in the thousands place and an 8 in the ten thousands place.

Eighty-five thousand, six hundred fifty-three is represented numerically as 85,653.

One million is a 1 followed by six 0s. Compare that to one googol, which is a 1 followed by one hundred 0s. The giant number was the inspiration behind "Google," the popular search engine that indexes a massive amount of digital information.

1.9 Write the value numerically: four million, five hundred two thousand, nine hundred.

Begin at the right side of the phrase, with the words "nine hundred." This indicates a 9 belongs in the hundreds place. Because you are given no information about the ones or tens place, both should be occupied by 0s. Moving left, the phrase "five hundred two thousand" provides values for the hundred thousands place (5) and the thousands place (2), but no information about the ten thousands place is given, so another 0 should be used. Finally, moving left across the second comma in the phrase, the digit 4 is presented as the value for the millions place.

Four million, five hundred two thousand, nine hundred is represented numerically as 4,502,900.

1.10 Write the value numerically: thirty-five billion, seventy million, twenty.

Only four place values are provided in this problem: "twenty" indicates a 2 in the tens place; "seventy million" indicates a 7 in the ten millions place; and the phrase "thirty five billion" provides the values for two digits—5 in the billions place and 3 in the ten billions place.

Thirty-five billion, seventy million, twenty is represented numerically as 35,070,000,020.

One billion = 1,000,000,000. It has TWO more sets of triple zeroes than one thousand (1,000), which is why it's called one billion. ("Bi" is a prefix that means "two.") One trillion has THREE more sets of triple zeroes than one thousand (1,000,000,000,000). Quadrillions have FOUR times as many, and quintillions have FIVE times as many.

Rounding and Estimating

Accuracy to Specific Place Values

1.11 Round 29 to the nearest ten.

In order to round a number to a specific place value, first identify the digit immediately to the right of the place to which you are rounding. If the digit is 4 or less, change it to a 0 and do nothing else. If, however, the digit you identified is 5 or greater, change that digit to a 0 and add 1 to the left digit.

In other words, if the digit is 0, 1, 2, 3, or 4.

In this problem, you are asked to round to the nearest ten. The number 29 has a 2 in the tens place, so you investigate the digit 9, which is immediately to the right of the tens place. Because 9 is greater than or equal to 5, change it to a 0 and add 1 to the value in the tens place: $2 + 1 = 3$.

The number 29, rounded to the nearest ten, is 30.

1.12 Round 73 to the nearest ten.

You are asked to round to the tens place; this number contains a 7 in the tens place, so your ultimate goal is to decide whether to leave 7 in the tens place or add 1 to it, changing it to 8. This decision is made by considering the digit immediately to the right of 7, the value in the ones place: 3. Because 3 is less than 5, change the digit 3 into 0 and do nothing else.

The number 73, rounded to the nearest ten, is 70.

Rounding a number in the 70s to the nearest ten will either give you 70 or 80 as an answer. You're basically asking this question: Is the number closer to 70 or to 80? If the number is 74 or less, it's closer to 70. If the number is 75 or more, it's closer to 80.

1.13 Round 482 to the nearest hundred.

The problem directs you to round to the nearest hundred. This number has a 4 in the hundreds place, so consider the digit immediately to the right of 4—in this case, 8. Because 8 is greater than or equal to 5, you must add 1 to the digit in the hundreds place $(4 + 1 = 5)$.

The number 482, rounded to the nearest hundred, is 500. Notice that a rounded number always has zeroes for *each* of the place values to the right of the digit you are rounding. In this problem, you are rounding to the nearest hundred, so the tens and ones places both contain zeroes.

1.14 Round 2,984 to the nearest hundred.

You are instructed to round to the hundreds place, and in this number, the hundreds place is occupied by a 9. Consider the digit to its right (in the tens place), which is 8. Because the tens digit is greater than or equal to 5, you should add 1 to the hundreds place.

This poses a dilemma, because adding 1 to the 9 in the hundreds place causes a one-digit number (9) to change into a two-digit number $(9 + 1 = 10)$. You cannot squeeze both digits of the sum 10 into the hundreds place. Instead, place the right digit of 10—which is 0—into the hundreds place and add the left digit—which is 1—to the thousands place. The number 2,984, rounded to the nearest hundred, is 3,000.

This only happens when the place you are rounding to is 9 and you are forced to round up because the number to the right is greater than or equal to 5.

If these steps are counterintuitive, disregard the rote technique often applied to rounding problems, step back, and see the proverbial forest for the trees. Rounding to the nearest hundred means providing an answer ending in "00," just as the number one hundred (100) ends in "00." The problem asks this question: "Is 2,984 closer to 2,900 or 3,000?" In this problem, rounding to the nearest hundred is the same as rounding to the nearest thousand.

1.15 Round 2,783 to the nearest: (a) ten, (b) hundred, and (c) thousand.

This problem asks:
Is 2,783 is closer to ...
(a) 2,780 or 2,790?
(b) 2,700 or 2,800?
(c) 2,000 or 3,000?

(a) Identify the digit to the right of the tens place. In this case, that digit is 3. Because 3 is less than five, you should change it to 0 and leave the digit in the tens place unchanged. The number 2,783, rounded to the nearest ten, is 2,780.

(b) Identify the digit to the right of the hundreds place; in this number, that digit is 8. Because 8 is greater than or equal to 5, change 8 and the numbers to its right to zeroes and add 1 to the hundreds place. The number 2,783, rounded to the nearest hundred, is 2,800.

(c) Identify the digit to the right of the thousands place, in this case 7. Because 7 is greater than or equal to 5, change it and the digits to its right into zeroes and add 1 to the digit in the thousands place. The number 2,783, rounded to the nearest thousand, is 3,000.

1.16 Round 18,495 to the nearest: (a) ten, (b) hundred, and (c) thousand.

(a) The ones digit (5) is greater than or equal to 5, so add 1 to the tens digit. Notice that the tens digit is 9, so you should change it to 0 and add 1 to the hundreds digit instead. The number 18,495, rounded to the nearest ten, is 18,500.

This is very similar to Problem 1.14. When you round up a 9, make it a 0 and add one to the digit on its left instead.

(b) The tens digit (9) is greater than or equal to 5, so add 1 to the hundreds digit. The number 18,495, rounded to the nearest hundred, is 18,500.

(c) The hundreds digit (4) is less than 5, so change it and the digits to its right into zeroes, leaving the thousands digit alone. The number 18,495, rounded to the nearest thousand, is 18,000.

1.17 Round 419,085 to the nearest: (a) hundred, (b) thousand, and (c) ten thousand.

(a) To round to the nearest hundred, inspect the digit in the tens place (in this case, 8). Because 8 is greater than or equal to 5, you should add 1 to the hundreds digit. The number 419,085, rounded to the nearest hundred, is 419,100.

(b) To round to the nearest thousand, inspect the digit in the hundreds place. Because 0 is less than 5, the thousands digit is unchanged. The number 419,085, rounded to the nearest thousand, is 419,000.

(c) To round to the nearest ten thousand, inspect the digit in the thousands place. Because 9 is greater than or equal to 5, you should add 1 to the ten thousands place. The number 419,085, rounded to the nearest ten thousand, is 420,000.

Make sure you start with the original number 419,085 at the beginning of each part of this problem. Don't use an already rounded version from another part.

1.18 Estimate the sum 68 + 25 by rounding each of the numbers to the nearest ten.

When a precise answer is not required, you can round the numbers in a problem in order to calculate a relatively close approximation, or estimate, of the answer. (The mathematics symbol ≈ is used in place of the familiar equal sign "=" to indicate that values are approximately equal, rather than exactly equal.)

This problem directs you to round each of the numbers to the nearest ten: $68 \approx 70$ and $25 \approx 30$. Rewrite the addition problem using the rounded values.

$$68 + 25 \approx 70 + 30$$

While it is not difficult to add 68 and 25, it is arguably quicker to add 70 + 30 because you do not have to contend with any digits in the ones place of either number. Notice that $70 + 30 = 100$. Therefore, $68 + 25 \approx 100$.

1.19 Estimate the sum 492 + 385 by rounding each of the numbers to the nearest hundred.

Round each of the numbers to the nearest hundred: $492 \approx 500$ and $385 \approx 400$. Replace the numbers in the addition problem with the rounded values.

$$492 + 385 \approx 500 + 400$$

Instead of adding 492 + 385, you're adding 500 + 400, which is no harder than adding 5 + 4. 5 hundreds plus 4 hundreds equals 9 hundreds.

Because $500 + 400 = 900$, you conclude that $492 + 385 \approx 900$.

1.20 Estimate the sum 16,214 + 3,548 by rounding each of the numbers to the nearest thousand.

Round each of the numbers to the nearest thousand: $16{,}214 \approx 16{,}000$ and $3{,}548 \approx 4{,}000$. Add the rounded values: $16{,}000 + 4{,}000 = 20{,}000$. You conclude that $16{,}214 + 3{,}548 \approx 20{,}000$.

Number Classifications

Even, odd, natural, whole, integer, rational, irrational, real

1.21 Describe, and give two examples of, odd numbers.

If a number contains a 1, 3, 5, 7, or 9 in the ones place, it is considered an odd number. Regardless of how large or how small the number is—or how many digits the number contains—only the digit in the ones place determines whether or not a number is odd. For example, the number 27 is odd because the ones digit is 7. Similarly, the number 2,842,603 is odd because the digit in the ones place is 3.

1.22 Describe, and give two examples of, even numbers.

Even numbers can be divided by two and not leave a remainder. If an odd number is divided by two, the remainder will always be 1. Look for more information about division and remainders in Problems 2.26–2.34.

According to Problem 1.21, odd numbers contain a 1, 3, 5, 7, or 9 in the ones place. Even numbers, on the other hand, have either 0, 2, 4, 6, or 8 in the ones place. For example, the number 30 is even because it contains 0 in the ones place. Similarly, the number 13,598 is even because it contains 8 in the ones place.

1.23 Identify the set of numbers classified as "natural numbers."

The natural numbers are the set of numbers beginning with one and counting upwards. For this reason, they are also called the "counting numbers":

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \ldots$$

1.24 Identify the set of numbers classified as "whole numbers."

The whole numbers include all of the natural numbers (described in Problem 1.23) and the number zero:

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots$$

Some books define "whole numbers" as any number that isn't a fraction or decimal—even if it's negative.

The inclusion of zero as the defining difference between the natural and whole numbers represents the general consensus among mathematicians, but that distinction is not absolute mathematical canon. Some mathematicians and texts make no distinction between natural, counting, and whole numbers; they use the words interchangeably to describe the set of numbers listed above.

1.25 Describe, and give two examples of, negative numbers.

Positive numbers either have a "+" sign in front of them or no sign at all. All of the numbers in Problems 1.1–1.24 have been positive numbers.

Negative numbers are values less than zero and are identified by the symbol "–" that immediately precedes them. For instance, the number –2 (read "negative two") is negative. Although Chapter 3 will feature a more in-depth discussion of positive and negative numbers, you likely have an innate understanding of negative numbers already. For example, if a meteorologist is forecasting cold temperatures for your region, he may express "ten degrees below zero" as the negative number –10.

1.26 Describe, and give two examples of, integers.

Integers are positive and negative numbers that do not contain fractional or decimal portions. Any whole number, like 3, is an integer, as is its negative (or opposite). Therefore, –3 is also an integer. The integers are best described as the set of numbers listed below, noting that "…" means the numbers continue infinitely in each direction:

$$\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots$$

1.27 Describe, and give two examples of, rational numbers.

Rational numbers are values that can be expressed as fractions. Any fraction, whether positive or negative, is considered a rational number. Thus, $\frac{3}{4}$ and $-\frac{1}{5}$ are both rational numbers.

This book won't really explore fractions until Chapter 7.

Although rational numbers may be expressed as fractions, they may also be expressed in other forms as well. See Problem 1.29 for more information.

1.28 What are the primary characteristics of a number expressed as a decimal?

A number expressed as a decimal contains the symbol "." called the decimal point. A number may have digits on both sides of the decimal point, but the decimal point is usually omitted if the numbers to its right are all zeroes.

Consider the decimal 152.94. The digit immediately left of the decimal point (in this case, 2) represents the ones place. Therefore, 5 occupies the tens place and 1 occupies the hundreds place. The digits right of the decimal place are similarly named, but they contain the suffix "-ths." In the decimal 152.94, 9 occupies the tenths place and 4 occupies the hundredths place. A more in-depth discussion of decimals begins in Chapter 10.

There is no oneths place. The first digit to the right of the decimal point represents the tenths place.

1.29 Identify the two circumstances under which decimal numbers are also rational numbers, and give an example of each.

If a decimal does not have infinitely many digits to the right of a decimal point, it is called a "terminating decimal." For instance, 0.25 and 6.1775 are both terminating decimals, as 0.25 contains two digits after the decimal point and 6.1775 contains four digits after the decimal point. Because the decimal portions of these two numbers do not continue infinitely, they are both rational numbers, which means they can both be represented as fractions. (The technique used to transform decimals into fractions is presented in Problems 11.1–11.15.)

If the decimal portion of a number does not terminate but does repeat itself in a predictable pattern, then that number is also rational. For example, the decimal 0.3333333333…, which contains the digit 3 in each of its decimal places, is a rational number. That number may be written as $0.\overline{3}$; a bar is drawn over 3 to indicate that the digit repeats infinitely.

By definition, if a number is rational, you must be able to express it as a fraction. If you transform that fraction into a decimal, then the decimal will either terminate or infinitely repeat. As a final example, consider the fraction $\frac{7}{13}$. The decimal form of this fraction is $0.538461538461538 46\ldots$. The notation "..." at the end of the decimal indicates that it continues infinitely (and therefore does not terminate). However, the digits "538461" repeat, so you could write $\frac{7}{13}=0.\overline{538461}$.

1.30 Given the decimal form of a number, explain how you can determine whether or not that number is irrational.

According to Problem 1.29, the decimal portion of a rational number either terminates or infinitely repeats. The mathematical opposite of "rational" is "irrational," so by definition, an irrational number has a decimal portion that neither terminates nor repeats itself. Consider the decimal representation of the irrational number $\sqrt{3}$:

$\sqrt{3}$ is read "square root of three." Aside from a quick mention here, you won't hear about square roots again until Chapter 14.

$$\sqrt{3}=1.7320508075688772935274463415058\ldots$$

Although certain digits repeat (the decimal contains "88," "77," and "44"), the entire decimal does not settle into a repetitive pattern, no matter how many digits you list to the right of the decimal point. Therefore, $\sqrt{3}$ is an irrational number.

1.31 What two number classifications, when combined, comprise the real numbers? Use your answer to construct a rudimentary definition of real numbers.

To construct the set of real numbers, combine the set of all rational numbers with the set of all irrational numbers. Any real number, therefore, is either a rational or an irrational number. According to Problems 1.29 and 1.30, which define rational and irrational numbers, a real number is a decimal that terminates, continues infinitely and repeats, or continues infinitely but does not repeat. Any conceivable decimal meets one of those conditions, so the most basic definition of a real number is "any number than can be expressed as a decimal."

Common Math Symbols

A guide to the non-numeric symbols you'll see

Problem 1.18 also discussed these symbols, and ≈ is used in Problems 1.18–1.20.

1.32 Identify the mathematical symbols that represent "equal to" and "approximately equal to." Use them to write the sentences below mathematically.

Forty is equal to forty. Forty-one is approximately equal to forty.

The symbol "=" joins two equal values, whereas the symbol "≈" joins two approximately equal values. Use = to rewrite the left sentence and ≈ to rewrite the right sentence.

$$40 = 40 \qquad 41 \approx 40$$

1.33 Identify the mathematical symbol that represents "not equal to" and use it to rewrite the sentence below mathematically.

Four is not equal to seven.

You can also use a slash to make the symbols "not greater than," "not less than," "not greater than or equal to" and "not less than or equal to."

To construct the symbol for "not equal to," draw a slash through an equality symbol: ≠. The "not equal to" symbol is the first of five inequality symbols introduced in this section of the book. The other four inequality symbols are introduced in Problems 1.37–1.38.

$$4 \neq 7$$

1.34 Identify the mathematical symbols that represent the operations of addition and subtraction and use them to rewrite the sentences below.

The sum of two and six is equal to eight. The difference of ten and three is equal to seven.

The symbol "+" represents addition, and the symbol "–" represents subtraction. There are two key words in the given sentences that indicate which symbol should be used. When two values are combined through addition, the result is called a "sum"; when one value is subtracted from another, the result is called the "difference."

$$2 + 6 = 8 \qquad 10 - 3 = 7$$

1.35 Identify the three notation styles most commonly used to represent multiplication and rewrite the sentence below using each notation.

Two multiplied by five is equal to ten.

In elementary mathematics, the symbol "×" is used to represent multiplication: $2 \times 5 = 10$. However, the variable x often used in algebra looks very similar to the multiplicative symbol ×, so two alternate annotation techniques are used in more advanced mathematics: parentheses and a multiplication dot.

Sometimes multiplication needs no symbol at all. The expression ab means "a times b."

A multiplication dot indicates multiplication when it replaces the "×" symbol: $2 \cdot 5 = 10$. Similarly, values separated by parentheses are also multiplied: $2(5) = 10$.

1.36 Identify the three notation styles most commonly used to represent division and rewrite the sentence below using each notation.

Eight divided by four is equal to two.

The traditional division sign, first used by primary students, is consistently used in intermediate and advanced mathematics: $8 \div 4 = 2$. A slash may be used to indicate division, though it is less common: $8 / 4 = 2$. Fractions, by definition, also express division: $\frac{8}{4} = 2$.

1.37 Identify the mathematical symbols that represent "less than" and "less than or equal to." Use them to rewrite the sentences below mathematically.

One is less than three. Four is less than or equal to five.

Any number is less than OR EQUAL TO itself because every number is equal to itself. For the same reason, any number is also greater than or equal to itself.

The inequality symbol "<" represents "less than"; the value on the left side of the symbol must be less than the value on the right side. Placing a line segment beneath the symbol transforms it into "≤," which translates "less than or equal to."

$$1 < 3 \qquad 4 \le 5$$

1.38 Identify the mathematical symbols that represent "greater than" and "greater than or equal to." Use them to rewrite the sentences below mathematically.

Ten is greater than nine. Two is greater than or equal to two.

The bottom of the symbol has two lines, so it almost looks like an equal sign down there. Some people write the "or equal to" inequality symbols like this: $\geqq$ $\leqq$. Those look a little more like equal signs.

"Greater than" is represented by the inequality symbol ">"; adding a line segment beneath the symbol (like in Problem 1.37) changes its meaning to "greater than or equal to": ≥.

$$10 > 9 \qquad 2 \ge 2$$

1.39 Explain how to round 52 to the nearest ten, using an inequality symbol in your explanation.

In order to round a number to the nearest ten, you must consider the digit in the ones place. In this instance, that digit is 2. Because $2 < 5$, you should change the digit 2 to a 0 and leave the digit in the tens place unaltered. The number 52, rounded to the nearest ten, is 50.

1.40 Explain how to round 153 to the nearest hundred, using an inequality symbol in your explanation.

In order to round a number to the nearest hundred, you should consider the digit in the tens place. If it is greater than or equal to five, you must add one to the digit in the hundreds place.

In this problem, a 5 appears in the tens digit. Because $5 \geq 5$, you should add one to the hundreds digit: $1 + 1 = 2$. Therefore, the number 153, rounded to the nearest hundred, is 200.

Chapter 2

BASIC BINARY OPERATIONS

+, −, ·, and ÷

Binary operations are algorithms, represented by a symbol, that produce a value given two input values. The most basic binary operation is addition, represented by the addition sign "+"; two numbers connected by an addition sign should be combined into a sum. Similarly, the binary operations of subtraction, multiplication, and division produce differences, products, and quotients, respectively, each time given two input values. The chapter concludes with common axiomatic assumptions that govern binary operations.

Yikes. That paragraph is a mouthful. Here's what you need to know: This chapter helps you review how to add, subtract, multiply, and divide. Those four calculations are called binary operations because you start with two ("bi-" means two) numbers, one on each side of the symbols you're probably familiar with (+, −, ·, and ÷). The fancy word for the result you get when you add numbers is called the sum. Each operation has its own fancy term: difference = subtraction, product = multiplication, and quotient = division.

At the end of the chapter, things get a little theoretical. The book introduces axioms (which are also called properties) of binary operations. They are mathematical rules that explain commonsense rules like "you can add numbers in any order and still get the same answer." That one's called the commutative property of addition.

Addition

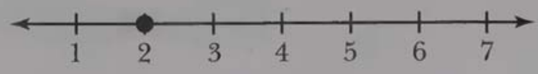
Combine numbers to get a sum

So it's a line with numbers on it. Makes sense.

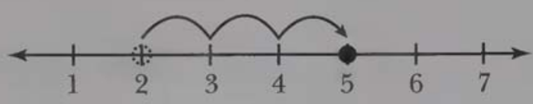
2.1 Use a number line to calculate the sum: $2 + 3$.

A number line is a line with equally spaced segments that represent numeric values. Consider the number line below, on which the natural numbers are evenly spaced. The first number in the addition problem $2 + 3$ is marked with a dot.

If you don't know what the natural numbers are, flip to Problem 1.23.



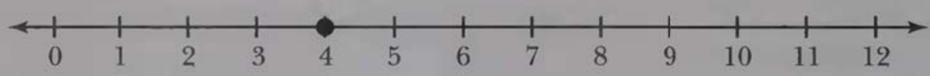
The problem instructs you to add 3 to the number 2. Therefore, you should move three places to the right of the dot you placed at 2, as illustrated in the following figure.



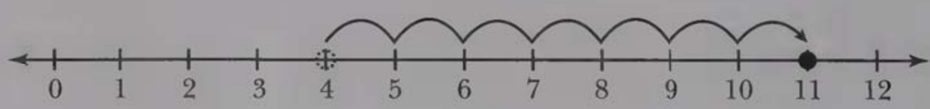
The new location of the dot identifies the sum: $2 + 3 = 5$.

2.2 Use a number line to calculate the sum: $4 + 7$.

Draw a number line and place a dot at 4, the first number in the addition problem.



The second number in the addition problem is 7, so move seven places to the right of the dot you placed at 4.



The final location of the dot represents the sum of the numbers: $4 + 7 = 11$.

2.3 Write the addition problem vertically and calculate the sum: 35 + 41.

Once numbers are sufficiently large, adding them using a number line (like in Problems 2.1–2.2) is neither easy nor helpful. Instead, you should list the numbers vertically, as demonstrated below.

$$\begin{array}{r} 35 \\ +41 \\ \hline \end{array}$$

Line up the numbers so the ones places are in the right column and the tens places are in the left column.

Begin with the right column, which contains the ones digits for each number: 5 and 1. Add the digits together and write the sum below the horizontal line.

$$\begin{array}{r} 35 \\ +41 \\ \hline 6 \end{array}$$

Now add the digits in the tens places (3 + 4 = 7) and write the sum below the horizontal line beneath the column of tens digits.

$$\begin{array}{r} 35 \\ +41 \\ \hline 76 \end{array}$$

The two-digit sum appears below the horizontal line: 35 + 41 = 76.

2.4 Write the addition problem vertically and calculate the sum: 17 + 25.

Write the number 25 below the number 17 so that the ones and tens places align.

$$\begin{array}{r} 17 \\ +25 \\ \hline \end{array}$$

Add the digits in the ones column: 7 + 5 = 12. This sum contains two digits, but you should only write the ones digit of the sum (2) below the vertical line. The tens digit of the sum (1) should be written above the tens digits in the left column.

If you add one-digit numbers and get a two-digit number, only put the right digit of the answer below the line. The left digit appears as a tiny number at the top of the left column. This is usually called "carrying the one."

$$\begin{array}{r} {}^{1} \\ 17 \\ +25 \\ \hline 2 \end{array}$$

Add the numbers in the tens column, including the digit 1 produced when calculating the sum in the ones column: $1 + 1 + 2 = 4$.

$$\begin{array}{r} {}^{1} \\ 17 \\ +25 \\ \hline 42 \end{array}$$

Thus, $17 + 25 = 42$.

2.5 Write the addition problem vertically and calculate the sum: $68 + 12$.

Write the numbers 68 and 12 so that the tens and ones places are in distinct columns.

$$\begin{array}{r} 68 \\ \underline{+12} \end{array}$$

Add the numbers in the ones column: $8 + 2 = 10$. As demonstrated in Problem 2.4, you should write the ones digit (0) of the sum below the horizontal line in the ones column and carry the tens digit (1) to the top of the tens column.

$$\begin{array}{r} {}^{1} \\ 68 \\ +12 \\ \hline 0 \end{array}$$

Add the digits in the tens column, including the 1 carried from the previous step.

$$\begin{array}{r} {}^{1} \\ 68 \\ +12 \\ \hline 80 \end{array}$$

The sum of 68 and 12 is 80.

This time the digits in the tens column add up to a two-digit number. Put the right digit below the line and the left digit at the top of the column to the left (in this case, the hundreds place column).

2.6 Write the addition problem vertically and calculate the sum: $251 + 364$.

Write the numbers 251 and 364 so that the digits in the hundreds, tens, and ones places align. Then, add the columns of digits from right to left, beginning with the ones place: $1 + 4 = 5$. Notice that the sum of the digits in the tens place is a two-digit number: $5 + 6 = 11$. You should carry the tens digit (1) of the sum to the top of the hundreds column and add it to that sum.

$$\begin{array}{cccc} & {}^{1} & & \\ & 2 & 5 & 1 \\ + & 3 & 6 & 4 \\ \hline & 6 & 1 & 5 \end{array}$$

Thus, $251 + 364 = 615$.

2.7 Write the addition problem vertically and calculate the sum: 4,938 + 2,615.

Write the problem vertically and add the ones digits: 8 + 5 = 13. Place the 3 below the line in the ones column and carry the tens digit to the top of the next column.

$$\begin{array}{rrrrr} & & & {\scriptstyle 1} & \\ & 4, & 9 & 3 & 8 \\ + & 2, & 6 & 1 & 5 \\ \hline & & & & 3 \end{array}$$

Add the digits in the tens column, including the digit carried from the previous step: 1 + 3 + 1 = 5.

$$\begin{array}{rrrrr} & & & {\scriptstyle 1} & \\ & 4, & 9 & 3 & 8 \\ + & 2, & 6 & 1 & 5 \\ \hline & & & 5 & 3 \end{array}$$

Add the digits in the hundreds column: 9 + 6 = 15. Place the ones digit of the sum (5) below the line and place the tens digit (1) above the numbers in the next column.

$$\begin{array}{rrrrr} & {\scriptstyle 1} & & {\scriptstyle 1} & \\ & 4, & 9 & 3 & 8 \\ + & 2, & 6 & 1 & 5 \\ \hline & & 5 & 5 & 3 \end{array}$$

Add the digits in the thousands column: 1 + 4 + 2 = 7.

$$\begin{array}{rrrrr} & {\scriptstyle 1} & & {\scriptstyle 1} & \\ & 4, & 9 & 3 & 8 \\ + & 2, & 6 & 1 & 5 \\ \hline & 7, & 5 & 5 & 3 \end{array}$$

The sum of 4,938 and 2,615 is 7,553.

2.8 Write the addition problem vertically and calculate the sum: 6,587 + 3,994.

Each column of digits has a two-digit sum, so you will need to carry the digit 1 to the top of each column as you proceed from right to left.

$$\begin{array}{rrrrr} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & \\ & 6, & 5 & 8 & 7 \\ + & 3, & 9 & 9 & 4 \\ \hline & 10, & 5 & 8 & 1 \end{array}$$

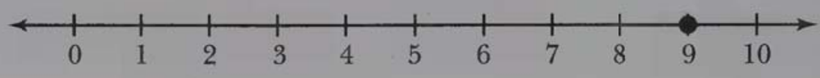
Subtraction

Find the difference between numbers

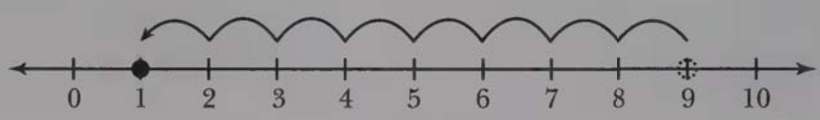
2.9 Use a number line to calculate the difference: $9 - 8$.

Draw a dot on a number line that marks the first number in the subtraction problem: 9. Note that the order in which subtraction takes place affects the answer, so make sure to place a dot on the number appearing left of the subtraction sign.

In addition, the order doesn't matter: $2 + 3$ and $3 + 2$ both equal 5. That's almost never true in subtraction: $6 - 2$ is not the same as $2 - 6$.



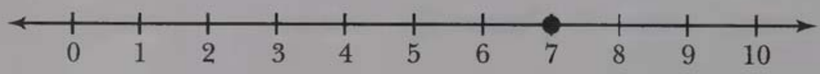
Problems 2.1–2.2 explain how to add numbers using a number line: A dot representing one number is moved to the right, according to the value of the second number. Addition and subtraction are opposite operations, so to subtract a value, you should move *left*. In this problem, you are subtracting 8 from 9, so you should move the dot eight spaces to the left.



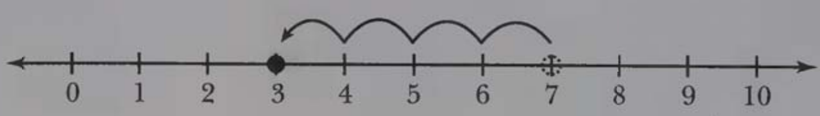
The final location of the dot represents the difference, so $9 - 8 = 1$.

2.10 Use a number line to calculate the difference: $7 - 4$.

Place a dot on a number line representing the value left of the subtraction sign.



You are subtracting 4 from 7, so move the dot 4 spaces to the left of 7 along the number line.



The difference $7 - 4$ is equal to 3.

2.11 Write the subtraction problem vertically and calculate the difference: 85 – 21.

Write the number 21 below the number 85 so that the ones and tens places of the numbers align.

$$\begin{array}{r} 85 \\ \underline{-\,21} \end{array}$$

Subtract the digits in the ones column ($5 - 1 = 4$) and write the answer below the horizontal line in the ones column.

$$\begin{array}{r} 85 \\ \underline{-\,21} \\ 4 \end{array}$$

Now subtract the digits in the tens column ($8 - 2 = 6$) and write the answer below the horizontal line once again, this time in the tens column.

$$\begin{array}{r} 85 \\ \underline{-\,21} \\ 64 \end{array}$$

In other words, make sure 3 goes under the 6, because they are both in the ones place.

Therefore, $85 - 21 = 64$.

2.12 Write the subtraction problem vertically and calculate the difference: 16 – 3.

Write the number 3 below the number 16, ensuring that the corresponding place values align.

$$\begin{array}{r} 16 \\ \underline{-\;\,3} \end{array}$$

Subtract the digits in the ones column: $6 - 3 = 3$.

$$\begin{array}{r} 16 \\ \underline{-\;\,3} \\ 3 \end{array}$$

Now subtract the digits in the tens place. Note that the bottom number, 3, does not have a tens digit, so imagine that the missing tens digit is 0. The difference of the tens digits is $1 - 0 = 1$.

You don't have to write this zero in here—people usually leave it out. Just don't forget that you need to subtract in the tens digits, even if you don't write the zero.

$$\begin{array}{r} 16 \\ \underline{-\,03} \\ 13 \end{array}$$

Thus, $16 - 3 = 13$.

2.13 Write the subtraction problem vertically and calculate the difference: 34 – 9.

Write 9 below 34, ensuring that the digits in the ones places (4 and 9) are aligned in the right column.

$$\begin{array}{r} 34 \\ -\ 9 \\ \hline \end{array}$$

You can take 9 away from 34 without getting a negative number—you just can't take 9 away from 4. Time for a new technique: borrowing.

Problems 2.11–2.12 direct you to subtract the digits in the ones column, but you cannot: 4 – 9 would be a negative number. In order to calculate the difference, you must first "borrow" from the tens place. Subtract one from the tens digit in the top number (3 – 1 = 2) and place it next to the 4 in the ones place.

$$\begin{array}{r} {}^{2}\cancel{3}\,{}^{1}4 \\ -\quad 9 \\ \hline \end{array}$$

Now that 4 has been transformed into 14 (by adding the 10 you borrowed from the tens place), you can calculate 14 – 9 = 5.

$$\begin{array}{r} {}^{2}\cancel{3}\,{}^{1}4 \\ -\quad 9 \\ \hline 5 \end{array}$$

Subtract the digits in the tens place. Remember that you subtracted 1 from 3. Thus, you are subtracting 0 (there is no tens digit in the bottom number) from 2.

$$\begin{array}{r} {}^{2}\cancel{3}\,{}^{1}4 \\ -\quad 9 \\ \hline 2\ \ 5 \end{array}$$

The difference of 34 and 9 is 25.

2.14 Write the subtraction problem vertically and calculate the difference: 62 – 48. Use addition to check your answer.

Write 48 below 62 so that the ones and tens places of each number align.

$$\begin{array}{r} 62 \\ -\ 48 \\ \hline \end{array}$$

Notice that you cannot subtract the digits in the ones column, so you must borrow from the tens column. Subtract one from the tens place of 62 and write it in front of the ones place.

In Problem 2.4, you "carried the 1." Now you're doing the opposite, "borrowing the 1."

$$\begin{array}{r} {}^{5}\cancel{6}\,{}^{1}2 \\ -\;\;4\;\;8 \\ \hline \end{array}$$

Calculate the difference in the ones column $(12 - 8 = 4)$ and the tens column $(5 - 4 = 1)$.

$$\begin{array}{r} {}^{5}\cancel{6}\,{}^{1}2 \\ -\;\;4\;\;8 \\ \hline 1\;\;4 \end{array}$$

Thus, $62 - 48 = 14$. To check your answer, add the difference (14) to the smaller number in the subtraction problem (48). The sum should be the larger number (62).

$$\begin{array}{r} {}^{1}\;\; \\ 14 \\ +48 \\ \hline 62 \end{array}$$

2.15 Write the subtraction problem vertically and calculate the difference: $513 - 247$.

Express the difference vertically and borrow from the tens place to subtract the digits in the ones place.

$$\begin{array}{r} 5\;\;{}^{0}\cancel{1}\;\;{}^{1}3 \\ -\;\;2\;\;\;4\;\;\;7 \\ \hline 6 \end{array}$$

To subtract the digits in the tens column, you must once again borrow a digit, this time from the hundreds place. Subtract 1 from the hundreds digit of the top number $(5 - 1 = 4)$ and place the 1 in front of the tens digit in the top number. Calculate the difference in the tens column.

The tens digit of the top number is 0 at the moment—you had to subtract 1 from it in the first step. When you put the 1 in front of it that you're borrowing from the hundreds column, you end up with 10 in the tens column.

$$\begin{array}{r} {}^{4}\cancel{5}\;\;{}^{10}\cancel{1}\;\;{}^{1}3 \\ -\;\;2\;\;\;\;4\;\;\;7 \\ \hline 6\;\;\;6 \end{array}$$

Finally, subtract the digits in the hundreds column.

If you want to, you can check your answer by adding 266 and 247. The sum is 513 (the original number you subtracted from) so you got the problem right.

$$\begin{array}{rccc} & {}^{4}\cancel{5} & {}^{10}\cancel{1} & {}^{1}3 \\ - & 2 & 4 & 7 \\ \hline & 2 & 6 & 6 \end{array}$$

Thus, 513 – 247 = 266.

2.16 Write the subtraction problem vertically and calculate the difference: 3,602 – 895.

Align the digits, ensuring that corresponding place values appear in the same columns.

$$\begin{array}{r} 3,602 \\ -\ \ 895 \\ \hline \end{array}$$

You cannot subtract the digits in the ones column because $2 < 5$. However, you cannot borrow a digit from the tens place of 3,602, because that digit is 0. So that you may *eventually* borrow from the tens column, you must first borrow a digit from the hundreds column and place it before the digit in the tens column.

Subtract 1 from the 6 in the hundreds column of 3,602 $(6 - 1 = 5)$ and place that 1 in front of the 0 in the tens column.

$$\begin{array}{rcccc} & 3, & {}^{5}\cancel{6} & \cancel{10} & 2 \\ - & & 8 & 9 & 5 \\ \hline \end{array}$$

Now that the tens place of the top number contains something other than zero, you can borrow from it to subtract the digits in the ones column.

Subtract 1 from 10 to get 9 and place that 1 in front of the 2. Subtract: $12 - 5 = 7$.

$$\begin{array}{rcccc} & 3, & {}^{5}\cancel{6} & {}^{9}\cancel{10} & {}^{1}2 \\ - & & 8 & 9 & 5 \\ \hline & & & & 7 \end{array}$$

Both of the digits in the tens column are 9. Calculate the difference: $9 - 9 = 0$.

$$\begin{array}{rcccc} & 3, & {}^{5}\cancel{6} & {}^{9}\cancel{10} & {}^{1}2 \\ - & & 8 & 9 & 5 \\ \hline & & & 0 & 7 \end{array}$$

Borrow 1 from the thousands column to subtract the digits in the hundreds column and complete the problem.

$$\begin{array}{rcccc} & {}^{2}\cancel{3}, & {}^{15}\cancel{6} & {}^{9}\cancel{10} & {}^{1}2 \\ - & & 8 & 9 & 5 \\ \hline & 2, & 7 & 0 & 7 \end{array}$$

Thus, 3,602 – 895 = 2,707.

2.17 Write the subtraction problem vertically and calculate the difference: 5,000 – 743.

In order to subtract the ones digits, you must borrow from the tens digit of 5,000. However, like Problem 2.16, that digit is 0. When you attempt to borrow from the hundreds place instead, that digit is 0 as well. Therefore, you must borrow from the thousands place in order to change the hundreds digit of the top number into something other than zero.

$$\begin{array}{rrrrr} & {}^{4}\cancel{5}, & 10 & 0 & 0 \\ - & & 7 & 4 & 3 \\ \hline \end{array}$$

Borrow from the hundreds place to change the tens place into a value other than zero.

$$\begin{array}{rrrrr} & {}^{4}\cancel{5}, & {}^{9}\cancel{10} & 10 & 0 \\ - & & 7 & 4 & 3 \\ \hline \end{array}$$

Finally, borrow from the tens place to change the ones place into a value other than zero.

$$\begin{array}{rrrrr} & {}^{4}\cancel{5}, & {}^{9}\cancel{10} & {}^{9}\cancel{10} & 10 \\ - & & 7 & 4 & 3 \\ \hline \end{array}$$

Subtract the digits in each column to calculate the difference.

$$\begin{array}{rrrrr} & {}^{4}\cancel{5}, & {}^{9}\cancel{10} & {}^{9}\cancel{10} & 10 \\ - & & 7 & 4 & 3 \\ \hline & 4, & 2 & 5 & 7 \end{array}$$

Therefore, 5,000 – 743 = 4,257.

Multiplication

Add a group of numbers repeatedly

This product (multiplication problem) could also be written 2×6 OR $2(6)$.

2.18 Explain how the operation of multiplication derives values by repeatedly adding groups of numbers. Use the product $2 \cdot 6$ as an example.

The operation of multiplication is, at its core, repetitive addition. Consider the product $2 \cdot 6$. To calculate this value, imagine 2 groups of 6 objects, as illustrated below.

◊◊◊◊◊◊ ◊◊◊◊◊◊

The total number of objects in the two groups is 12, so $2 \cdot 6 = 12$.

2.19 Calculate $4 \cdot 5$ using the method described in Problem 2.18 and verify that $4 \cdot 5 = 5 \cdot 4$.

Represent the product $4 \cdot 5$ visually by drawing four groups of five objects, as illustrated below.

••••• ••••• ••••• •••••

You can reverse the numbers when you add or multiply and it doesn't affect the answer. See Problem 2.38 for more information.

There are 20 total objects, so $4 \cdot 5 = 20$. Now represent the product $5 \cdot 4$ visually, as five groups of four objects.

•••• •••• •••• •••• ••••

Once again, there are 20 objects, so the products $5 \cdot 4$ and $4 \cdot 5$ are both equal to 20.

2.20 Write the multiplication problem vertically and calculate the product: $13 \cdot 2$.

Write the number 2 beneath 13 so that the ones places of both numbers align in a column. Note that multiplication problems written vertically often use the × symbol, instead of the · symbol.

$$\begin{array}{r} 13 \\ \underline{\times\ 2} \end{array}$$

If you don't know that $3 \cdot 2 = 6$, you should memorize the multiplication tables at the end of the book in Appendix A.

Multiply the digits in the ones column: $3 \cdot 2 = 6$. Write the product below the horizontal line in the ones column.

$$\begin{array}{r} 13 \\ \underline{\times\ 2} \\ 6 \end{array}$$

Now multiply the bottom digit in the product (2) by the digit in the tens place of the top number (1): $2 \cdot 1 = 2$. Record the result below the horizontal line beneath the tens digit.

$$\begin{array}{r} 13 \\ \underline{\times\ 2} \\ 26 \end{array}$$

The product of 13 and 2 is 26. Unlike addition and subtraction, in which you only perform operations on digits in the same columns, multiplication requires you to multiply each digit of the top number by each digit of the bottom number. In this case, you multiplied the single digit of the bottom number (2) by each digit of the top number (13), one at a time.

2.21 Write the multiplication problem vertically and calculate the product: $6 \cdot 15$.

Align the ones digits of the numbers and write the smaller number beneath the larger number.

$$\begin{array}{r} 15 \\ \times\ 6 \\ \hline \end{array}$$

Multiply the digits in the ones column: $5 \cdot 6 = 30$. Because this product is larger than one digit, record only the right digit of 30 beneath the horizontal line; write the tens digit above 1, the tens digit of the top number.

$$\begin{array}{r} {\scriptstyle 3} \\ 15 \\ \times\ 6 \\ \hline 0 \end{array}$$

Now multiply the digit directly above the horizontal line (6) by the tens digit of 15: $6 \cdot 1 = 6$. Before you record this value below the horizontal line, add the small number 3 that resulted from the previous step: $6 + 3 = 9$.

$$\begin{array}{r} {\scriptstyle 3} \\ 15 \\ \times\ 6 \\ \hline 90 \end{array}$$

Thus, $6 \cdot 15 = 90$.

2.22 Write the multiplication problem vertically and calculate the product: $283 \cdot 7$.

Write 7 below 283, aligning the digits in the ones place. Your goal is to multiply 7 by each of the digits of 283, one at a time, from right to left. The first product is $7 \cdot 3 = 21$. Write the ones digit 1 below the horizontal line and the tens digit 2 above the tens digit of 283.

$$\begin{array}{r} {\scriptstyle 2} \\ 283 \\ \times\ \ 7 \\ \hline 1 \end{array}$$

Now multiply 7 by the tens digit of 283 and add the 2 that resulted from the previous step: $7 \cdot 8 = 56$, and $56 + 2 = 58$.

$$\begin{array}{r} {\scriptstyle 5\,2} \\ 283 \\ \times\ \ 7 \\ \hline 81 \end{array}$$

Finally, multiply the hundreds digit of 283 by 7 and add the tens digit 5 from the previous step: $7 \cdot 2 = 14$, and $14 + 5 = 19$.

$$\begin{array}{r} {\scriptstyle 52} \\ 283 \\ \times \ \ 7 \\ \hline 1{,}981 \end{array}$$

Thus, $283 \cdot 7 = 1{,}981$.

2.23 Write the multiplication problem vertically and calculate the product: $45 \cdot 18$.

Both of these numbers have two digits, unlike the last couple of multiplication problems, where you multiplied by a one-digit number.

Align the digits so that the tens places—as well as the ones places—align. Begin with the ones digit of 18 and multiply it by each of the digits in 45, from right to left.

$$\begin{array}{r} {\scriptstyle 4} \\ 45 \\ \times 18 \\ \hline 360 \end{array}$$

Before you multiply the tens digit of 18 by the digits of 45, add a place holder 0 below 360. Erase the small 4 above the problem that was used in the previous step.

You need this zero because you're multiplying by a digit in the tens place now. If you were multiplying by a digit in the hundreds place, you'd need a placeholder of "00."

$$\begin{array}{r} 45 \\ \times 18 \\ \hline 360 \\ 0 \end{array}$$

Now multiply the tens digit of 18 by each digit of 45, from right to left: $1 \cdot 5 = 5$ and $1 \cdot 4 = 4$. Record the products next to the placeholder digit 0.

$$\begin{array}{r} 45 \\ \times 18 \\ \hline 360 \\ 450 \end{array}$$

Add the numbers below the horizontal line.

$$\begin{array}{r} 45 \\ \times 18 \\ \hline {\scriptstyle 1} \\ 360 \\ +450 \\ \hline 810 \end{array}$$

The product of 45 and 18 is 810.

2.24 Write the multiplication problem vertically and calculate the product: $609 \cdot 23$.

Multiply the ones digit of 23 by each digit of 609, from right to left.

$$\begin{array}{r} \scriptstyle 2 \\ 609 \\ \times \quad 23 \\ \hline 1{,}827 \end{array}$$

3 · 9 = 27, so 7 goes below the line and 2 floats above the 0 in 609.

3 · 0 = 0, and 0 + tiny floating 2 = 2

3 · 6 = 18

Include a place holder digit 0 below the ones place of 1,827. Then, multiply the tens digit of 23 by each digit of 609, from right to left.

$$\begin{array}{r} 609 \\ \times \quad 23 \\ \hline 1{,}827 \\ +12{,}180 \\ \hline \end{array}$$

Record the sum of 1,827 and 12,180 below the horizontal line.

$$\begin{array}{r} 609 \\ \times \quad 23 \\ \hline \scriptstyle 1\ 1 \quad \\ 1{,}827 \\ +12{,}180 \\ \hline 14{,}007 \end{array}$$

Thus, $609 \cdot 23 = 14{,}007$.

2.25 Write the multiplication problem vertically and calculate the product: $726 \cdot 453$.

Multiply each digit of 453, one at a time, by each digit of 726, from right to left. Remember to include a place holder digit 0 when multiplying by the tens digit of 453 and place holder digits 00 when multiplying by the hundreds digit of 453.

$$\begin{array}{r} 726 \\ \times \quad 453 \\ \hline 2{,}178 \\ 36{,}300 \\ +290{,}400 \\ \hline 328{,}878 \end{array}$$

0 and 00 are the place holders. The next line, if there were one, would have a place holder of 000.

Division

Separate a number into smaller groups

"Distinct" means the groups don't overlap. Every object can only belong to one group.

2.26 Solve the division problem $12 \div 3$ visually by dividing 12 objects into distinct groups, each containing 3 objects.

Consider the below illustration of 12 objects.

◊◊◊◊◊◊◊◊◊◊◊◊

Use slashes to separate the group of 12 objects into smaller groups, each containing three objects.

◊◊◊/◊◊◊/◊◊◊/◊◊◊

The larger group of 12 objects divides into four groups of three objects. Therefore, $12 \div 3 = 4$.

2.27 Explain how to check your solution to Problem 2.26.

Multiplication and division are opposite operations, much like addition and subtraction. In Problem 2.14, addition is used to check the answer to a subtraction problem; similarly, you use multiplication to check answers to division problems.

According to Problem 2.26, $12 \div 3 = 4$. To check the quotient, multiply it (4) and the number by which you were dividing (3). The answer should be the number that was divided into groups (12).

$$4 \cdot 3 = 12$$

2.28 Use the method described in Problem 2.26 to identify the quotient and remainder of the division problem and check your answer: $21 \div 8$.

Draw a group of 21 objects.

•••••••••••••••••••••

Beginning on the left, count groups of eight objects and separate them from the larger group using slashes.

••••••••/••••••••/•••••

You are able to form two groups of eight objects, so the quotient is 2. However, you are left with a partial group of five objects in the rightmost group. That number is called the remainder. The answer is written "$21 \div 8 = 2$ r 5," and is read "Twenty-one divided by eight is equal to two with a remainder of five."

The number you're dividing by.

To check your answer, multiply the quotient (2) and the divisor (8). Then, add the remainder (5). The result should be the dividend (21).

The number you're dividing into.

$$\begin{aligned}(\text{quotient} \cdot \text{divisor}) + \text{remainder} &= \text{dividend} \\ (2 \cdot 8) + 5 &= 21 \\ 16 + 5 &= 21 \\ 21 &= 21\end{aligned}$$

2.29 Calculate the quotient: $78 \div 2$.

To apply the method described in Problem 2.26, you would need to draw 78 objects and divide them into groups of 2. Whereas this would produce the correct answer, the process itself is not viable as the numbers in the division problems increase in size and complexity. Instead, you should apply the process often called "long division" by writing the divisor and dividend as illustrated below.

$$2\overline{)78}$$

Divide the first digit of the dividend (7) by the divisor. Because you are now dealing with much smaller numbers, you can apply the technique described in Problem 2.26 if you wish. The result is 3 r 1. Write the digit above 7, as illustrated below.

You could also ask yourself "What's the biggest number that, when multiplied by 2, gives a result no bigger than 7?" The answer is 3.

$$\begin{array}{r} 3 \\ 2\overline{)78} \end{array}$$

Multiply 3 by the divisor 2 and write the product ($3 \cdot 2 = 6$) below 7. Then, subtract 6 from 7 as illustrated below. Note that the difference $7 - 6 = 1$ is equal to the remainder you calculated in the previous step.

$$\begin{array}{r} 3 \\ 2\overline{)\ 78} \\ \underline{-6} \\ 1 \end{array}$$

You are now finished with the first digit of the dividend. Drop the second digit (8) down so it shares the same line with 1, the difference you calculated in the previous step.

$$\begin{array}{r} 3 \\ 2\overline{)\ 78} \\ \underline{-6\downarrow} \\ 18 \end{array}$$

Divide 18 by the divisor 2. Notice that there is no remainder this time: $18 \div 2 = 9$. Write 9 at the top of the problem above the digit 8 you placed next to the difference in the previous step.

$$\begin{array}{r} 39 \\ 2\overline{)\ 78} \\ \underline{-6} \\ 18 \end{array}$$

Multiply the new digit of the quotient (9) by the divisor and write the result $(9 \cdot 2 = 18)$ below 18. When you subtract, the difference is 0, which again matches the remainder calculated in the preceding step.

$$\begin{array}{r} 39 \\ 2\overline{)\ 78} \\ \underline{-6} \\ 18 \\ \underline{-18} \\ 0 \end{array}$$

There are no digits left in the dividend, so the problem is complete. The quotient is the number above the division symbol: $78 \div 2 = 39$.

2.30 Calculate the quotient: $49 \div 3$.

Prepare the problem for long division.

$$3\overline{)49}$$

The dividend is the big number, 49. So $4 \div 3 = 1$ remainder 1.

Dividing the first digit of the dividend by the divisor produces a quotient of 1 r 1. Write the quotient above the division symbol, multiply by the divisor $(1 \cdot 3 = 3)$ and subtract the result from the first digit.

$$\begin{array}{r} 1 \\ 3\overline{)\ 49} \\ \underline{-3} \\ 1 \end{array}$$

Drop the next digit of the dividend to the lowest line.

$$\begin{array}{r} 1 \\ 3\overline{)\ 49} \\ \underline{-3\downarrow} \\ 19 \end{array}$$

Now calculate $19 \div 3$. This time, the numbers do not divide evenly—a remainder is left: $19 \div 3 = 6 \text{ r } 1$.

$$\begin{array}{r} 16 \\ 3\overline{)\ 49} \\ \underline{-3} \\ 19 \\ \underline{-18} \\ 1 \end{array}$$

No digits are left in the dividend to drop down and place next to 1, the difference calculated in the previous step. Therefore, the problem is complete and the number left at the bottom (1) is the remainder. You conclude that $49 \div 3 = 16 \text{ r } 1$.

2.31 Calculate the quotient and check your answer: $194 \div 8$.

Rewrite the expression as a long division problem.

$$8\overline{)194}$$

The first digit of the dividend (1) is less than the divisor (8), so rather than divide only the first digit by 8, you should divide the first *two* digits of the dividend: $19 \div 8 = 2 \text{ r } 3$.

When the number you're dividing by is greater than the first digit of the number you're dividing into, divide into the first TWO digits instead. In this case, $19 \div 8$ instead of $1 \div 8$.

$$\begin{array}{r} 2 \\ 8\overline{)\ 194} \\ \underline{-16} \\ 3 \end{array}$$

Note that you write the digit 2 above 9 rather than above 1 in the quotient. This indicates that you divided 8 into 19 rather than dividing 8 into 1. Drop the next digit to the lowest line of the problem and divide by the divisor: $34 \div 8 = 4 \text{ r } 2$.

$$\begin{array}{r} 24 \\ 8\overline{)\ 194} \\ \underline{-16} \\ 34 \\ \underline{-32} \\ 2 \end{array}$$

Thus, $194 \div 8 = 24 \text{ r } 2$. Check your answer by multiplying the quotient and the divisor and then adding the remainder to the result.

$$
\begin{aligned}
(\text{quotient} \cdot \text{divisor}) + \text{remainder} &= \text{dividend} \\
(24)(8) + 2 &= 194 \\
192 + 2 &= 194 \\
194 &= 194
\end{aligned}
$$

The result, 194, is equal to the original dividend, so the answer is correct.

2.32 Calculate the quotient: $287 \div 25$.

Prepare the problem using long division notation.

$$25\overline{)287}$$

If the first two digits of the dividend were 21, you'd have a problem, because 21 < 25. In that case, you'd have to include another digit and divide 25 into ALL THREE digits of 287.

This problem has a two-digit divisor, so you should divide it into the first *two* digits of the dividend: $28 \div 25 = 1$ r 3. Write the quotient above the second digit of 28 to indicate that you divided into the first two digits of the dividend.

$$\begin{array}{r} 1 \\ 25\overline{)287} \end{array}$$

Write the product of 1 and 25 below the dividend, and subtract it from 28.

$$\begin{array}{r} 1 \\ 25\overline{)\;287} \\ \underline{-25} \\ 3 \end{array}$$

Drop the final digit of the dividend to the lowest line and repeat the process: $37 \div 25 = 1$ r 12.

$$\begin{array}{r} 11 \\ 25\overline{)\;287} \\ \underline{-25} \\ 37 \\ \underline{-25} \\ 12 \end{array}$$

Therefore, $287 \div 25 = 11$ r 12.

Note: Problems 2.33–2.34 refer to the following division problem: 415 ÷ 12.

2.33 A student attempting to solve this problem begins with the steps illustrated below. Identify the error in the incomplete work.

$$\begin{array}{r} 2 \\ 12\overline{)\,415} \\ \underline{-24} \\ 17 \end{array}$$

The student placed a 2 above 41 in the dividend and then correctly multiplied that number by the divisor 12 to get 24. Next, 24 is subtracted from 41 to get a correct result of 17. However, this difference presents a red flag; it is too large.

17 > 12

Whenever a difference in a long division problem is greater than the divisor, you did not place a large enough digit above the dividend. In this case, the number 2 above 41 should have been 3, because 41 ÷ 12 = 3 r 5, and the remainder (5) is now less than the divisor (12).

As the number of digits in the divisor grows, so does the difficulty choosing the correct number to place above the dividend. Many students use trial and error to identify the correct number. They select a digit, multiply it by the divisor, and calculate the difference. If the difference is larger than the divisor, as it was in this incorrect work sample, they choose a larger number and repeat the process.

Note: Problems 2.33–2.34 refer to the following division problem: 415 ÷ 12.

2.34 Calculate the quotient, correcting the error identified in Problem 2.33.

According to Problem 2.33, the digit 3 should replace 2 above the dividend.

$$\begin{array}{r} 3 \\ 12\overline{)\,415} \\ \underline{-36} \\ 5 \end{array}$$

Drop the remaining digit of the dividend to the lowest line and notice that 55 ÷ 12 = 4 r 7.

$$\begin{array}{r} 34 \\ 12\overline{)\,415} \\ \underline{-36} \\ 55 \\ \underline{-48} \\ 7 \end{array}$$

Therefore, 415 ÷ 12 = 34 r 7.

2.35 Calculate the quotient: 1,074 ÷ 21.

The first two digits of the dividend form a number that is less than the divisor, so you should divide 21 into the first three digits of 1,074: 107.

10 < 21

$$\begin{array}{r} 5 \\ 21\overline{)\,1,074} \\ -\,105 \\ \hline 2 \end{array}$$

Drop the final digit of the dividend to the lowest line and complete the problem: 24 ÷ 21 = 1 r 3.

$$\begin{array}{r} 51 \\ 21\overline{)\,1,074} \\ -\,105 \\ \hline 24 \\ -21 \\ \hline 3 \end{array}$$

Thus, 1,074 ÷ 21 = 51 r 3.

Fundamental Properties

"Common Sense" rules

2.36 Identify the algebraic property that guarantees (2 + 9) + 6 = 2 + (9 + 6) and verify that the statement is true.

There's a specific order for operations that's more detailed than "parentheses first." This is covered in Problems 4.29–4.40.

An operation in parentheses should be completed before other operations. According to the associative property of addition, the manner in which a set of numbers is grouped does not affect the sum of those numbers.

You add the numbers 2, 9, and 6 in a different order on each side of the equal sign. Even though the numbers are written in the same order, they are grouped differently. On the left side of the equation, 2 and 9 appear within parentheses and should be added first.

$$\begin{aligned} &(2+9)+6 \\ &=11+6 \\ &=17 \end{aligned}$$

Compare this result to the addition problem on the right side of the equation, in which 9 and 6 appear in parentheses.

$$\begin{aligned}&2+(9+6)\\&=2+15\\&=17\end{aligned}$$

Both expressions produce the same sum, illustrating the associative property of addition.

Think of it this way: Numbers in parentheses associate with each other more closely, but no matter where the parentheses are and how the numbers associate, it doesn't change the final sum.

2.37 Identify the algebraic property that guarantees $(3 \cdot 2) \cdot 5 = 3 \cdot (2 \cdot 5)$ and verify that the statement is true.

In Problem 2.36, two identical sets of numbers are added—the only difference between the expressions is the order in which they are added, based on the presence of parentheses. According to the associative property, both of the sums are equal. Similarly, in this problem, two identical sets of numbers are multiplied, and the only difference between the sets is the order of multiplication dictated by the parentheses.

On the left side of the equation, you must calculate $3 \cdot 2$ first and then multiply the product by 5; on the right side of the equation, you calculate $2 \cdot 5$ first and then multiply that product by 3. Notice that both produce the same final product of the three numbers, as demonstrated below.

$$\begin{aligned}(3\cdot2)\cdot5&=3\cdot(2\cdot5)\\6\cdot5&=3\cdot10\\30&=30\end{aligned}$$

The associative property of multiplication states that numbers have the same product regardless of the way in which they are grouped by parentheses.

There is no associative property for subtraction or division. There's also no commutative property for subtraction or division (see Problem 2.38).

2.38 Describe the commutative properties of addition and multiplication and give an example of each.

The commutative property states that the order in which an operation is completed has no effect on the sum or product. Given the numbers 2 and 7, the sum is always 9, regardless of which number is written first.

$$2+7=9 \qquad 7+2=9$$

Similarly, the product of 2 and 7 is 14, regardless of the order in which the multiplication is expressed.

$$2\cdot7=14 \qquad 7\cdot2=14$$

The commutative property applies neither to subtraction nor to division. Consider the differences and quotients below, which change when the numbers are reversed.

$$\begin{aligned}4-1&\neq1-4 \qquad & 10\div2&\neq2\div10\\3&\neq-3 & 5&\neq0.2\end{aligned}$$

Any number minus itself equals zero. Technically, subtracting is just adding a negative number, so that's why a property involving subtraction is called the additive inverse property.

2.39 Describe the additive inverse property and demonstrate it using the number 10.

According to the additive inverse property, the sum of every number and its opposite is zero. For instance $10 - 10 = 0$.

2.40 Describe the multiplicative inverse property and demonstrate it using the number 7.

There are a lot of words in this sentence that you learn later in the book. For now, think of it this way: Any number divided by itself is equal to 1.

The multiplicative inverse property states that the product of a number and its reciprocal is equal to one. For example, multiplying 7 by its reciprocal produces the following expression:

$$\frac{7}{7} = 1$$

Chapter 3

SIGNED NUMBERS

Positives and negatives

Chapters 1 and 2 introduce the concept of negative numbers, but only briefly. In Chapter 1, groups of numbers are defined, including integers, which may be positive or negative. At the end of Chapter 2, the additive inverse property is defined based on the sum of a number and its opposite—or negative—value. In this chapter, you will explore the concept of negative numbers in depth, and special attention is paid to the combination of positive and negative numbers via the binary operations presented in Chapter 2.

This chapter is all about signed numbers, which are (unsurprisingly) numbers that have signs—either positive or negative. Technically, all numbers have signs, but when you don't actually write a sign, you assume the number is positive. For example, most people write a positive 4 as "4" not "+4."

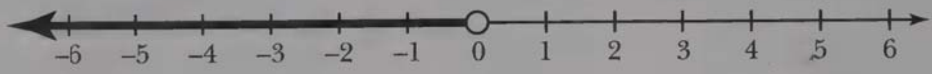
Chapter 2 is all about adding, subtracting, multiplying, and dividing numbers, and this chapter builds on that momentum. Most of the problems involve the same operations, but now some of the numbers are negative. You'll even see a few problems that deal with absolute values—grouping symbols that act like specialized parentheses but look like short vertical lines: | |.

Negatives and Opposites

What are negative numbers?

3.1 Draw a number line and darken the portion that represents the set of negative numbers.

A traditional number line representation of the real numbers places the positive numbers right of the number zero and negative numbers to the left of zero, as illustrated below. (Graphs like these will be investigated further in Problems 5.19–5.27.)

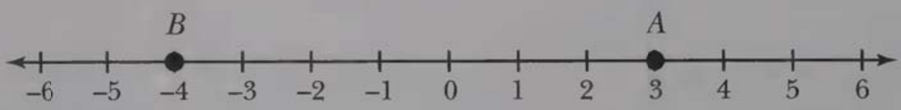


Zero is not a negative number, so you draw a hollow circle around it to say, "Every number left of—but not including—zero is negative."

All of the numbers along the shaded half of the number line are negative. Notice that an open point is placed at zero and acts as the upper boundary of the negative numbers. The circular boundary is an open (not solid) point because the boundary is not inclusive.

3.2 Plot the numbers $A = 3$ and $B = -4$ on a number line.

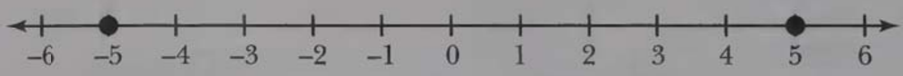
As illustrated below, $A = 3$ is a positive number, so it is three units right of zero along the number line. $B = -4$ is a negative number, so it is located four units left of zero. Place a solid point at each location and label each point with the corresponding letter.



3.3 Plot the number 5 and its opposite on a number line.

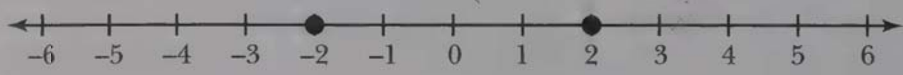
The opposite of a positive number is a negative number (and vice versa), so the opposite of 5 is –5.

Opposite numbers are the same distance from zero in opposite directions. The number 5 has no explicit sign, so 5 is understood to be positive and is located five units to the right of zero. Therefore, the opposite of 5 is five units left of zero: –5.



3.4 Plot the number –2 and its opposite on a number line.

Negative numbers are located left of zero on the number line, so plot –2 two units left of zero. Its opposite (2) is two units right of zero. Both numbers are illustrated below.



3.5 Translate the phrase below into a mathematical expression and identify its value.

The sum of 5 and the opposite of –8

The opposite of –8 is +8, so calculate the sum of 5 and 8.

$$5 + 8 = 13$$

You can write "the opposite of a number" using a negative sign. In other words, "the opposite of –8" could be written –(–8).

3.6 Translate the phrase below into a mathematical expression and identify its value.

2 less than the opposite of –6

The opposite of –6 is 6, so you are asked to identify the number that is 2 less than 6. The phrase "less than," in this problem, cues you to apply subtraction: $6 - 2 = 4$.

3.7 Translate the phrase below into a mathematical expression and identify its value.

The opposite of the product of 3 and 11

The opposite of a number is often written as that number preceded by a negative sign. In this case, the expression $-(3 \cdot 11)$ represents the opposite of the product $3 \cdot 11$. Thus, the opposite of the product of 3 and 11 is –33.

Absolute Values

A happy place, where everything is positive

3.8 Explain how to evaluate the following absolute value expressions: $|-10|$ and $|6|$.

The short vertical bars surrounding the numbers –10 and 6 are absolute value symbols. They are grouping symbols, like parentheses, but aside from merely grouping numbers, they also perform a specific role: They limit results to positive values.

Specifically, the absolute value of any negative number is the opposite, positive value of that number. In this problem, $|-10| = 10$. However, the absolute values do not indiscriminately change the signs of the numbers within. The absolute value of any positive number is that number, so $|6| = 6$.

Absolute values change negatives into positives and leave positives alone. Once you make the number inside positive, you stop writing the absolute value signs.

3.9 Simplify the expression: $12+|-4|$.

The number within the absolute value symbols is negative, so replace $|-4|$ with the opposite of –4, which is 4. Once you have replaced the absolute value expresssion, calculate the sum.

The absolute value of –4 is 4, so change –4 to 4 and drop the absolute value bars.

$$\begin{aligned} 12+|-4| &= 12+4 \\ &= 16 \end{aligned}$$

3.10 Simplify the expression: $16-|5|$.

The absolute value of any positive number is that positive number; in this case, $|5|=5$. Rewrite the expression and calculate the difference.

$$\begin{aligned} 16-|5| &= 16-5 \\ &= 11 \end{aligned}$$

3.11 Simplify the expression: $|-3|-|3|$.

Taking the absolute value of a number basically amounts to dropping the negative sign out front (if there is one).

This expression contains the absolute values of opposite numbers: –3 and 3. Note that both have the same value: $|-3|=3$ and $|3|=3$. Rewrite the expression replacing both absolute value expressions with the equivalent value of 3.

$$|-3|-|3|=3-3$$

Calculate the difference.

$$3-3=0$$

Thus, $|-3|-|3|=0$.

3.12 Simplify the expression: $|9-7|$.

Absolute value symbols, like parentheses, are grouping symbols, so any operations within absolute value bars should be completed before any other steps are taken. In this problem, the difference 9 – 7 appears within absolute value symbols, so you should begin by subtracting the values.

$$|9-7|=|2|$$

Complete the problem by simplifying the expression $|2|$. Note that the absolute value of any positive number is that positive number: $|2|=2$. Therefore, $|9-7|=2$.

3.13 Simplify the expression: $|-3||-5||-2|$.

This expression is a product of three absolute values. Note that each set of absolute values contains a negative number, so the absolute value of each is the opposite of that number.

$$|-3||-5||-2| = (3)(5)(2)$$

To calculate the product, multiply from left to right: $3 \cdot 5 = 15$, so replace $(3)(5)$ in the expression with 15.

$$(3)(5)(2) = (15)(2)$$

Finally, multiply $15 \cdot 2$ to complete the problem.

$$|-3||-5||-2| = 30$$

There are no operation symbols between the numbers (like +, –, or ·), so that means you're supposed to multiply them—after you calculate their absolute values, that is.

3.14 Simplify the expression: $14 + |-4| - |-3| + |8|$.

This expression contains the absolute values of two negative numbers and one positive number. Replace each with the appropriate absolute value: $|-4| = 4$, $|-3| = 3$, and $|8| = 8$.

$$14 + |-4| - |-3| + |8| = 14 + (4) - (3) + (8)$$

Add and subtract the numbers, working from left to right.

$$\begin{aligned} &= 18 - 3 + 8 \\ &= 15 + 8 \\ &= 23 \end{aligned}$$

Start by adding 14 + 4. Subtract 3 from that answer and then add 8.

Thus, $14 + |-4| - |-3| + |8| = 23$.

Sums and Differences of Signed Numbers

Add and subtract + and – numbers

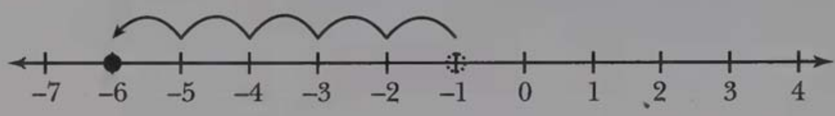
Note: Problems 3.15–3.16 refer to the following expression: –1 – 5.

3.15 Simplify the expression using a number line.

The expression –1 – 5 can be interpreted in two ways, both equally valid and both resulting in the same answer. You can choose to interpret the expression as a difference, subtracting 5 from the number –1. More commonly, however, the expression is interpreted as a combination of positive and negative numbers: Two negative numbers (–5 and –1) are combined to produce an even more negative number.

In other words, a number further to the left on the number line.

In the illustration below, a point is placed on a number line at –1, the first value in the expression when it is read from left to right. The value that follows is –5, so the point should be moved 5 units in the negative direction (left along the number line).



The combination of the negative numbers results in –6, so $-1 - 5 = -6$.

Note: Problems 3.15–3.16 refer to the following expression: –1 – 5.

3.16 Simplify the expression without the use of a number line.

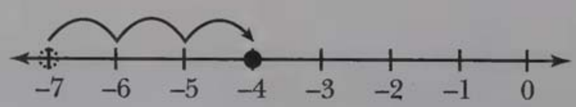
The expression consists of two numbers with the same sign—they are both negative. To combine the numbers, add their values while temporarily ignoring the shared negative sign ($1 + 5 = 6$) and then place the negative sign in front of the result: $-1 - 5 = -6$.

When combining two numbers in an expression, if the numbers share the same sign—whether positive or negative—you should add the numbers and then place the shared sign in front of the sum.

Note: Problems 3.17–3.18 refer to the following expression: –7 + 3.

3.17 Simplify the expression using a number line.

Begin on the left side of the expression and place a point on a number line at –7. The second number in the expression is +3, so move the point 3 units in the positive direction (to the right) on the number line, as illustrated below.



The final location of the point is –4, so you conclude that $-7 + 3 = -4$.

Note: Problems 3.17–3.18 refer to the following expression: –7 + 3.

3.18 Simplify the expression without the use of a number line.

This problem asks you to combine two numbers that have different signs—one is positive and one is negative. To do so, temporarily ignore the signs of the numbers and subtract the smaller number from the larger. In this case, if you omit the signs, you are left with the numbers 7 and 3. Calculate the difference: $7 - 3 = 4$.

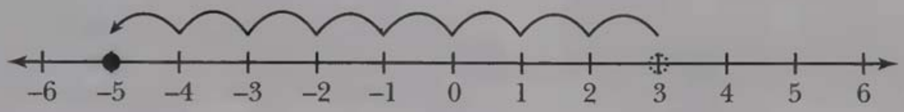
Now compare the two numbers you subtracted and identify the greater of the two. In this case, 7 is greater than 3. Refer back to the original problem and note that 7 is negative. Therefore, you should place a negative sign in front of the difference you calculated in the previous step: $-7 + 3 = -4$.

(1) Ignore the signs and subtract the numbers.
(2) Place the sign of the number you subtracted FROM in front of the result of Step (1).

Note: Problems 3.19–3.20 refer to the following expression: 3 – 8.

3.19 Simplify the expression using a number line.

To combine the numbers +3 and –8, place a point on the number line at 3 and then move 8 units in the negative direction, as illustrated below.



The final location of the point is 5 units left of 0, so $3 - 8 = -5$.

Note: Problems 3.19–3.20 refer to the following expression: 3 – 8.

3.20 Simplify the expression without the use of a number line.

The two numbers combined by this expression (3 and –8) have opposite signs. As explained in Problem 3.18, you should temporarily ignore the signs and subtract the smaller from the larger number: $8 - 3 = 5$. Next, compare the two numbers (while still ignoring the signs attached). Because $8 > 3$, you should place the sign associated with 8 in front of the difference you calculated in the previous step (5). Because 8 is negative, so is the answer: $3 - 8 = -5$.

–8 is more negative than +3 is positive, so when you combine the numbers, the answer has to be negative.

3.21 Simplify the expression: $-11 + 6 - 2$.

This expression directs you to combine three numbers. You accomplish this by combining two numbers at a time, working left to right. Start by combining –11 and +6. Because the numbers have opposite signs, you should calculate the difference of 11 and 6 and use the sign of the larger number. Therefore, $-11 + 6 = -5$. Rewrite the original expression, replacing $-11 + 6$ with the equivalent value –5.

11 > 6 and 11 is negative, so the difference is negative.

$$-11 + 6 - 2 = -5 - 2$$

Now combine the remaining numbers: –5 and –2. They both have the same sign, so you should add 2 and 5 and apply the shared negative sign to the sum.

$$-5 - 2 = -7$$

You conclude that $-11 + 6 - 2 = -7$.

3.22 Simplify the expression: $6 - 4 + 1 - 3$.

Combine the terms of the expression two at a time, working from left to right. In other words, calculate $6 - 4$, add 1 to the result, then subtract 3 from that result. These calculations are completed below, each on a separate line to clarify the arithmetic.

$$\begin{aligned}6-4+1-3&=2+1-3\\&=3-3\\&=0\end{aligned}$$

Thus, $6 - 4 + 1 - 3 = 0$.

3.23 Simplify the expression: $5 - (-9)$.

Two negative signs next to each other with no number in between.

This expression contains two adjacent signs. Before you can simplify the expression, you need to replace the "double signs" with a single sign according to the following criteria:

- If the adjacent signs are the same, they should be replaced with a single positive sign.
- If the adjacent signs are different, they should be replaced with a single negative sign.

This expression contains a negative sign adjacent to -9 in parentheses. Both signs are negative, so according to the rules above, the double negative signs should be replaced with a single positive sign.

Double negative signs turn into a positive sign (and so do double positive signs).

$$5 - (-9) = 5 + 9$$

To complete the problem, calculate the sum: $5 + 9 = 14$. You conclude that $5 - (-9) = 14$.

3.24 Simplify the expression: $-11 + (-7)$.

The expression contains adjacent signs + and – between -11 and 7. According to Problem 3.23, two adjacent signs that are different (one positive and one negative) should be replaced with a single negative sign.

$$-11 + (-7) = -11 - 7$$

Calculate the sum: $-11 - 7 = -18$. Thus, $-11 + (-7) = -18$.

3.25 Simplify the expression: $-2 + (-1) - (+5)$.

The expression contains two pairs of adjacent signs: + – before the middle term and – + before the final term. In both cases, the adjacent signs are different, so each pair should be replaced with a single negative sign.

$$-2 + (-1) - (+5) = -2 - 1 - 5$$

The result is an expression that combines three negative numbers. They all share a common sign, so you should calculate the sum of all three numbers and apply the shared negative sign.

$$-2-1-5=-8$$

Therefore, $-2+(-1)-(+5)=-8$.

3.26 Simplify the expression: $-10-(-6)+(-3)$.

The expression contains two pairs of adjacent signs: $--$ preceding the middle term and $+-$ preceding the last term. Pairs of like signs are replaced with a single positive sign and pairs of unlike signs are replaced with a single negative sign.

$$-10-(-6)+(-3)=-10+6-3$$

Combine the numbers in the new expression, working left to right.

$$\begin{aligned}-10+6-3&=-4-3\\&=-7\end{aligned}$$

Thus, $-10-(-6)+(-3)=-7$.

3.27 Simplify the expression: $14+|-8|-|+7|$.

Although this expression contains two pairs of adjacent signs, you cannot eliminate them using the technique described in Problem 3.23. Absolute values must be simplified before any other steps can be undertaken. Note that $|-8|=8$ and $|+7|=7$. Rewrite the expression, replacing the absolute values.

If you need to review absolute values, look at Problems 3.8–3.14.

$$14+|-8|-|+7|=14+(8)-(7)$$

Combine the resulting numbers to simplify the expression.

$$\begin{aligned}14+(8)-(7)&=22-7\\&=15\end{aligned}$$

Thus, $14+|-8|-|+7|=15$.

3.28 Simplify the expression: $|-3-9|-|-11-6|$.

Begin by simplifying the expressions within the absolute value symbols: $-3-9=-12$ and $-11-6=-17$.

$$|-3-9|-|-11-6|=|-12|-|-17|$$

Note the subtraction sign between the absolute values. It will remain after the absolute values of -12 and -17 are determined.

$$\begin{aligned}|-12|-|-17| &= 12-(17)\\ &= 12-17\\ &= -5\end{aligned}$$

You conclude that $|-3-9|-|-11-6|=-5$.

3.29 Simplify the expression: $16-(4+5)+|-3+2|$.

The parentheses and absolute value symbols are grouping symbols, so the expressions within should be simplified first: $4+5=9$ and $-3+2=-1$.

$$\begin{aligned}16-(4+5)+|-3+2| &= 16-(9)+|-1|\\ &= 16-9+|-1|\end{aligned}$$

The absolute value of a negative number is its opposite: $|-1|=1$. Substitute this value into the expression and combine the remaining numbers.

$$\begin{aligned}&= 16-9+(1)\\ &= 7+1\\ &= 8\end{aligned}$$

Products and Quotients of Signed Numbers

Multiply and divide + and – numbers

3.30 Is the following statement true or false? Explain your answer.

$$3\cdot 6=(-3)(-6)$$

This question asks whether the product of two numbers is equal to the product of their opposites, and the statement is true. Both products are equal to +18, according to the following rules:

- Two numbers with the same sign, whether positive or negative, have a positive product.
- Two numbers with different signs have a negative product.

The numbers +3 and +6 (on the left side of the equation) both share a positive sign, so the product is also positive: $3\cdot 6=18$. Similarly, –3 and –6 share a common sign, in this case a negative sign. Therefore, that product is positive as well: $(-3)(-6)=18$.

Most books use parentheses to indicate multiplication when negative numbers are involved: $-3\cdot -6$ looks weird with – and · right next to each other.

3.31 Simplify the expression: $2(-4)$.

Temporarily ignore the signs of both numbers and calculate the product: $2 \cdot 4 = 8$. Because 2 and -4 have opposite signs—one is positive and one is negative—the product is a negative number. Therefore, $2(-4) = -8$.

3.32 Simplify the expression: $(-10)(7)$.

Like the numbers in Problem 3.31, the two values in this expression have opposite signs. Therefore, the product of the numbers is negative: $(-10)(7) = -70$.

The signs of the numbers you're multiplying only affect the SIGN of your answer, not the answer itself: $(10)(7) = 70$ and $(-10)(7) = -70$.

3.33 Simplify the expression: $(-5)(-3)$.

According to Problem 3.30, two numbers with the same sign have a positive product. In this case, -5 and -3 are both negative, so they have the same sign. Therefore, $(-5)(-3) = 15$.

3.34 Simplify the expression: $(-2)(3)(-4)$.

Calculate the product of the first two numbers: $(-2)(3) = -6$. Rewrite the expression, replacing $(-2)(3)$ with the actual product.

$$(-2)(3)(-4) = (-6)(-4)$$

Complete the problem by calculating the product of the remaining negative numbers.

$$(-6)(-4) = 24$$

Therefore, $(-2)(3)(-4) = 24$.

When you're multiplying (or dividing) groups of numbers, count the negative signs. If that count is even, then the product is positive. If the count is odd, the product is negative.

3.35 Simplify the expression: $(-3)|-9-4|$.

This expression contains grouping symbols in the form of absolute values, so begin by simplifying the expression $-9 - 4$ within the absolute value symbols.

$$(-3)|-9-4| = (-3)|-13|$$

Note that the absolute value of -13 is 13.

$$(-3)|-13| = (-3)(13)$$

The product of a positive number and a negative number is negative.

$$(-3)(13) = -39$$

You conclude that $(-3)|-9-4| = -39$.

3.36 Simplify the expression: $-24 \div 2$.

The rules that govern the signs of products—stated in Problem 3.30—also apply to the signs of quotients:

- Two numbers with the same sign, whether positive or negative, have a positive quotient.
- Two numbers with different signs have a negative quotient.

If you need help with long division, review Problems 2.26–2.35.

This expression contains one positive and one negative number, so the quotient is negative. Use long division to divide the values; you have already determined that the answer will be negative, so there is no reason to include the negative sign in the long division problem.

$$\begin{array}{r} 12 \\ 2\overline{)\,24} \\ -2 \\ \hline 04 \\ -4 \\ \hline 0 \end{array}$$

Remember to apply a negative sign to the quotient: $-24 \div 2 = -12$.

3.37 Simplify the expression: $-153 \div (-7)$.

According to Problem 3.36, the quotient of two negative numbers is positive. Because the sign of the answer has been determined, omit the negative signs from the divisor and dividend.

$$\begin{array}{r} 21 \\ 7\overline{)\,153} \\ -14 \\ \hline 13 \\ -\ \ 7 \\ \hline 6 \end{array}$$

Thus, $-153 \div (-7) = 21 \text{ r } 6$.

3.38 Simplify the expression: $48 \div (-6) \div (-4)$.

If an expression contains a string of values that are multiplied and/or divided, like this expression, the operations should be carried out from left to right. Begin by calculating the quotient $48 \div (-6) = -8$. Notice that the answer is negative because the divisor and dividend have different signs. Substitute that value into the expression.

$$48 \div (-6) \div (-4) = -8 \div (-4)$$

Complete the problem by dividing –8 by –4. Because both numbers have the same sign, the quotient is positive.

$$-8 \div (-4) = 2$$

Therefore, $48 \div (-6) \div (-4) = 2$.

3.39 Simplify the expression: $-125 \div |-5|$.

Before you divide, simplify the absolute value expression: $|-5| = 5$.

$$-125 \div |-5| = -125 \div 5$$

Apply long division to calculate the quotient.

$$\begin{array}{r} 25 \\ 5\overline{)\ 125} \\ -10 \\ \hline 25 \\ -25 \\ \hline 0 \end{array}$$

The answer is negative because -125 and 5 have opposite signs.

You conclude that $-125 \div |-5| = -25$.

3.40 Simplify the expression: $-6 \div (-2) \cdot |-7|$.

According to Problem 3.38, multiplication and division should be completed from left to right, but before either operation is undertaken, you should simplify the absolute value in the expression.

$$-6 \div (-2) \cdot |-7| = -6 \div (-2) \cdot 7$$

Begin by dividing –6 by –2 and then multiply that result by 7.

$$\begin{aligned} -6 \div (-2) \cdot 7 &= 3 \cdot 7 \\ &= 21 \end{aligned}$$

Thus, $-6 \div (-2) \cdot |-7| = 21$.

3.41 Simplify the expression: $40 \div 4 \cdot (-2) \div (-5)$.

Work from left to right, multiplying or dividing only two numbers at a time.

$$\begin{aligned} 40 \div 4 \cdot (-2) \div (-5) &= 10 \cdot (-2) \div (-5) \\ &= -20 \div (-5) \\ &= 4 \end{aligned}$$

Chapter 4

ORDER OF OPERATIONS

What do you simplify first?

Unlike the English language, which is read from left to right, the order in which mathematics expressions are simplified is not dictated by the order in which the symbols appear, but rather the order of operations. In this chapter, you will explore grouping symbols and exponents and then apply those concepts to correctly simplify expressions of varying complexity based solely on the order of operations.

You've probably heard the mnemonic phrase "Please excuse my dear Aunt Sally." The first letter of each word represents a mathematical concept, and the order of the words provides the order in which you should simplify expressions: parentheses; exponents; multiplication and division; and addition and subtraction. That sequence is called the order of operations.

This chapter begins with a quick review of grouping symbols (like parentheses) [and brackets], which clump together pieces of an expression. It then moves on to exponents, the "excuse" of the Aunt Sally mnemonic phrase, takes a quick detour into the distributive property (a good example of the order of operations), and then lets you work with Aunt Sally herself, in all her arithmetic glory.

Grouping Symbols

The clumps in an expression

4.1 Verify that $16 - 5 + 7 \neq 16 - (5 + 7)$ and explain why the two expressions are unequal.

You can add and subtract numbers in order from left to right as long as the expression only contains addition and subtraction.

The expression $16 - 5 + 7$ on the left side of the "does not equal" sign is very similar to the expressions in Problems 3.21–3.22. To simplify the expression, combine the numbers working from left to right: Begin with 16, subtract 5, and then add 7 to the result.

$$\begin{aligned} 16-5+7 &= 11+7 \\ &= 18 \end{aligned}$$

You cannot add and subtract from left to right to simplify the expression $16 - (5 + 7)$ on the right side of the equation, because it contains grouping symbols in the form of parentheses that must be simplified first.

$$\begin{aligned} 16-(5+7) &= 16-(12) \\ &= 4 \end{aligned}$$

The expressions on both sides of the equation, which are the same except for a single pair of parentheses, produce different values. The grouping symbols on the right side cause you to subtract 12, instead of subtracting 5.

4.2 Verify that $5 \cdot 9 - 2 \neq 5 \cdot (9 - 2)$ and explain why the two expressions are unequal.

Like Problem 4.1, despite the same numbers, operations, and signs in the same order on both sides of the "does not equal sign," the grouping symbols on the right side will affect the result. Rather than multiplying 5 by 9 on the left side of the equation, the parentheses on the right side cause you to multiply 5 by 7. Verify that the expressions have unequal values by simplifying the expressions independently.

Treat the opposite sides of the equations as separate expressions.

$$\begin{gathered} 5 \cdot 9 - 2 \neq 5 \cdot (9-2) \\ 45 - 2 \neq 5 \cdot (7) \\ 43 \neq 35 \end{gathered}$$

4.3 Simplify the expression: $-2 \cdot [-18 \div 3]$.

The brackets in this expression, like the parentheses in Problems 4.1–4.2, are grouping symbols. The difference between parentheses and brackets is merely cosmetic—both serve the same purpose. Therefore, the expression within the brackets should be simplified first.

$$-2 \cdot [-18 \div 3] = -2 \cdot [-6]$$

Calculate the product of –2 and –6. Both numbers have the same sign, so the product is positive.

$$-2 \cdot [-6] = 12$$

Therefore, $-2 \cdot [-18 \div 3] = 12$.

4.4 Simplify the expression: $\frac{4+8}{-19+13}$.

A fraction consists of two expressions, the numerator, which appears above the fraction bar, and the denominator, which appears below the fraction bar. A fraction bar acts as a grouping symbol, separating the expressions in the numerator and denominator. Simplify both of the expressions independently.

$$\frac{4+8}{-19+13} = \frac{12}{-6}$$

Chapter 7 explains how to simplify fractions without having to turn them into division problems.

A fraction is also a quotient—the numerator of the fraction is divided by the denominator. To simplify the expression further, you should express it as a division problem and calculate the quotient.

$$\frac{12}{-6} = 12 \div (-6) = -2$$

You conclude that $\frac{4+8}{-19+13} = -2$.

4.5 Simplify the expression: $(-4 \cdot 7) - \frac{11+5}{2}$.

There are two different grouping symbols at work in this expression: a set of parentheses and a fraction. Multiply the numbers within the parentheses and add the numbers in the numerator of the fraction.

$$(-4 \cdot 7) - \frac{11+5}{2} = -28 - \frac{16}{2}$$

Simplify the fraction by expressing it as a quotient.

$$\begin{aligned} -28 - \frac{16}{2} &= -28 - (16 \div 2) \\ &= -28 - 8 \end{aligned}$$

Complete the problem by combining the negative numbers that remain.

$$-28 - 8 = -36$$

Thus, $(-4 \cdot 7) - \frac{11+5}{2} = -36$.

4.6 Simplify the expression: $4 \cdot [3-(5-9)]$.

The quantity $(5-9)$ is entirely contained (or "nested") within the larger quantity $[3-(5-9)]$. To simplify, begin with the innermost grouped expression and work outward. In this case, your first step is to calculate $5-9=-4$.

A "quantity" is an expression inside grouping symbols. For example 1 + (2 + 3) is read "1 plus the quantity 2 + 3."

$$4 \cdot [3-(5-9)] = 4 \cdot [3-(-4)]$$

Next, simplify the expression within brackets, the remaining grouping symbols in the problem. The expression $3-(-4)$ contains adjacent negative signs, which should be replaced with a positive sign.

$$\begin{aligned} 4 \cdot [3-(-4)] &= 4 \cdot [3+4] \\ &= 4 \cdot 7 \\ &= 28 \end{aligned}$$

3 - (-4) looks a lot like Problem 3.23, in case you need help with adjacent signs.

Therefore, $4 \cdot [3-(5-9)] = 28$.

4.7 Simplify the expression: $-6 \cdot [(-1-4)+|7-10|]$.

Like parentheses, brackets, and fractions, absolute value signs also act as grouping symbols. In this problem, two distinct quantities, $(-1-4)$ and $|7-10|$, are contained within a larger quantity that is bounded by brackets. They should be simplified first.

$$\begin{aligned} -6 \cdot [(-1-4)+|7-10|] &= -6 \cdot [(-5)+|-3|] \\ &= -6 \cdot [-5+3] \end{aligned}$$

Though you have now eliminated two grouped expressions, one remains; simplify it next.

$$\begin{aligned} -6 \cdot [-5+3] &= -6 \cdot [-2] \\ &= 12 \end{aligned}$$

4.8 Simplify the expression: $-20-[4 \cdot |-2+5|]$.

The expression $|-2+5|$ is nested within the larger quantity $[4 \cdot |-2+5|]$, so simplify the nested expression first.

$$\begin{aligned} -20-[4 \cdot |-2+5|] &= -20-[4 \cdot |3|] \\ &= -20-[4 \cdot 3] \end{aligned}$$

Next, simplify the expression within brackets, the only remaining grouping symbols.

$$\begin{aligned} -20-[4 \cdot 3] &= -20-[12] \\ &= -20-12 \\ &= -32 \end{aligned}$$

Distributive Property

Multiply a bunch of things by one number

4.9 According to the distributive property, $5(1+6) = 5(1) + 5(6)$. Verify that the statement is true.

5 times the quantity (1 + 6) is equal the sum of 5 times each of the things in parentheses.

The left expression consists of the parenthetical quantity $(1 + 6)$ multiplied by the number 5. According to the distributive property, you can "distribute" 5 through the parentheses—multiplying it by each of the numbers inside and then adding the results—and get the same answer.

Consider the left side of the equation. Simplifying it requires you to add the numbers within parentheses first.

$$\begin{aligned}5(1+6) &= 5(7)\\ &= 35\end{aligned}$$

Now simplify the right side of the equation to verify that it produces the same value, 35. Calculate the products $5(1)$ and $5(6)$ first and then add the results.

$$\begin{aligned}5(1)+5(6) &= 5+30\\ &= 35\end{aligned}$$

Note: Problems 4.10–4.11 refer to the following expression: 4(2 – 9 – 1).

4.10 Simplify the expression using the distributive property.

The distributive property only applies to quantities that contain addition and subtraction. For example, $2(5 \cdot 9) \neq 2(5) \cdot 2(9)$ because that quantity contains multiplication.

Distribute 4 through the quantity, multiplying it by every value individually.

$$4(2-9-1) = 4(2) + 4(-9) + 4(-1)$$

Calculate each of the products and simplify.

$$\begin{aligned}4(2)+4(-9)+4(-1) &= 8-36-4\\ &= -28-4\\ &= -32\end{aligned}$$

Note: Problems 4.10–4.11 refer to the following expression: 4(2 – 9 – 1).

4.11 Simplify the expression without using the distributive property.

Expressions within parentheses—or any grouping symbol—should be simplified before any other operations are completed. You should begin this problem by simplifying the parenthetical expression $2 - 9 - 1$.

$$\begin{aligned}4(2-9-1) &= 4(-7-1)\\ &= 4(-8)\\ &= -32\end{aligned}$$

Problems 4.10 and 4.11 both produce the same result: $4(2-9-1) = -32$.

4.12 Simplify using the following expression using the distributive property and check your answer: $-(-3+12)$.

Unlike Problem 4.10, which had a 4 outside the parentheses, this problem only has a negative sign.

The distributive property states that a number multiplied by a sum or difference can be multiplied by each of the numbers within that sum or difference. In this problem, only a negative sign appears outside the quantity. To apply the distributive property, treat the negative sign as if it were the number –1.

$$-(-3+12) = -1(-3+12)$$

Now that the quantity $(-3+12)$ is multiplied by a number—rather than just a negative sign—you can apply the distributive property. Multiply each of the values inside the quantity by –1 and combine the results.

$$\begin{aligned}-1(-3+12) &= (-1)(-3)+(-1)(12)\\ &= 3-12\\ &= -9\end{aligned}$$

To check your answer, simplify the original problem by combining the grouped numbers first. The answer should be the same: –9.

$$\begin{aligned}-(-3+12) &= -(9)\\ &= -9\end{aligned}$$

4.13 Simplify the expression using the distributive property: $2(3+7)+9(-12+8)$.

The book doesn't take an extra step to eliminate double signs from this point forward—it just changes them on the fly. Here are the rules one last time for reference:

$+ + = +$
$- - = +$
$+ - = -$
$- + = -$

Multiply both terms in the quantity $(3+7)$ by 2, and multiply both terms in the quantity $(-12+8)$ by 9. Both of the expressions may be expanded in the same step, as demonstrated below.

$$\begin{aligned}2(3+7)+9(-12+8) &= 2(3)+2(7)+9(-12)+9(8)\\ &= 6+14+(-108)+72\end{aligned}$$

The expression contains adjacent signs + and – between 14 and 108. Opposite adjacent signs are replaced by a single negative sign.

$$= 6+14-108+72$$

Combine the four remaining terms two at a time, working from left to right.

$$\begin{aligned}&= 20-108+72\\ &= -88+72\\ &= -16\end{aligned}$$

4.14 Simplify the expression using the distributive property: $6(-3-9)-5(4-3)$.

This problem is very similar to Problem 4.13 with one significant exception: The number distributed to the expression on the right is negative. Both of the terms in the quantity $(4-3)$ should be multiplied by -5.

$$\begin{aligned}6(-3-9)-5(4-3) &= 6(-3)+6(-9)-5(4)-5(-3)\\ &= -18-54-20+15\\ &= -72-20+15\\ &= -92+15\\ &= -77\end{aligned}$$

It would be much easier (and probably smarter) to simplify the grouped expressions first, but if you do, you won't end up distributing twice. Suppress the urge to simplify inside parentheses and brackets until the end.

4.15 Apply the distributive property twice to simplify the expression: $21-[-13+4(-2+12)]$.

This expression contains two quantities, one bounded by parentheses within a larger quantity bounded by brackets. According to Problem 4.6, the smaller quantity that appears within the larger quantity should be addressed first. Therefore, you should begin by distributing 4 to the terms within the parentheses.

$$\begin{aligned}21-[-13+4(-2+12)] &= 21-[-13+4(-2)+4(12)]\\ &= 21-[-13-8+48]\end{aligned}$$

The problem directs you to distribute twice, so rather than simplifying the terms within the brackets, you must distribute -1 to each of terms in the brackets.

Problem 4.12 explains why that − in front of the brackets should be treated like −1.

$$\begin{aligned}&= 21-1[-13-8+48]\\ &= 21-1(-13)-1(-8)-1(48)\\ &= 21+13+8-48\\ &= 34+8-48\\ &= 42-48\\ &= -6\end{aligned}$$

Exponents

Powers and repeated multiplication

4.16 Evaluate the expression by expanding it: 3^2.

You are given an exponential expression with base 3 raised to the second power. This notation indicates that two 3s should be multiplied.

$$\begin{aligned}3^2 &= 3\cdot 3\\ &= 9\end{aligned}$$

4.17 Evaluate the expression by expanding it: 4^3.

This exponential expression has a base of 4 and a power of 3. To evaluate the expression, multiply three 4s.

$$\begin{aligned} 4^3 &= 4\cdot4\cdot4 \\ &= 16\cdot4 \\ &= 64 \end{aligned}$$

4.18 Evaluate the expression by expanding it: 2^5.

Like Problems 4.16–4.18, this exponential expression is a shorthand method that indicates repeated multiplication. In this problem, a total of five 2s should be multiplied.

$$\begin{aligned} 2^5 &= 2\cdot2\cdot2\cdot2\cdot2 \\ &= 4\cdot2\cdot2\cdot2 \\ &= 8\cdot2\cdot2 \\ &= 16\cdot2 \\ &= 32 \end{aligned}$$

Raising something to the second power is called "squaring it." You can read $(-8)^2$ as "negative eight to the second power" or "negative eight squared."

4.19 Simplify the expression: $(-13+5)^2$.

Expressions within parentheses—or any grouping symbols—should be completed first when simplifying an expression. Combine the numbers –13 and 5 within the parenthetical quantity.

$$(-13+5)^2 = (-8)^2$$

Calculate the product of –8 and –8. Remember that the product of two negative numbers is a positive number.

$$\begin{aligned} (-8)^2 &= (-8)(-8) \\ &= 64 \end{aligned}$$

This rule seems arbitrary but it's not. If you're confused, flip ahead to Problem 4.29, which explains the exact order of operations explicitly.

4.20 Simplify the expression: $10^3 - 11^2$.

Exponential expressions should be evaluated before numbers are added, subtracted, multiplied, or divided—unless, of course, one of those four operations occurs within grouping symbols, because grouping symbols are always completed first.

$$\begin{aligned} 10^3 - 11^2 &= (10)(10)(10) - (11)(11) \\ &= (100)(10) - (11)(11) \\ &= 1{,}000 - (11)(11) \\ &= 1{,}000 - 121 \\ &= 879 \end{aligned}$$

4.21 Simplify the expression: $[-1-2]^{1+4}$.

Simplify the expression within brackets.

$$[-1-2]^{1+4}=[-3]^{1+4}$$

Next, simplify the expression in the exponential power.

$$[-3]^{1+4}=[-3]^5$$

Rewrite the exponential expression using repeated multiplication and calculate the product.

$$\begin{aligned}[-3]^5&=[-3][-3][-3][-3][-3]\\&=9[-3][-3][-3]\\&=-27[-3][-3]\\&=81[-3]\\&=-243\end{aligned}$$

4.22 Simplify the expression: $(6-8-5)^3$.

The expression consists of a quantity raised to an exponential power. Begin by combining the numbers within parentheses.

$$\begin{aligned}(6-8-5)^3&=(-2-5)^3\\&=(-7)^3\end{aligned}$$

Expand the expression, multiplying (–7) three times.

$$\begin{aligned}(-7)^3&=(-7)(-7)(-7)\\&=(49)(-7)\\&=-343\end{aligned}$$

Go from left to right: (–7)(–7) = +49 and then (+49)(–7) = –343.

4.23 Determine whether the following statement is true or false: $-4^2=(-4)^2$.

The equation is read, "The negative (or opposite) of four squared is equal to the quantity negative four squared." The statement is not true, as demonstrated below, because the left side instructs you to square 4, while the right side instructs you to square –4.

Look back at Problem 4.20: 10^3-11^2 simplifies to 1,000 – 121. You square 11, not –11, because the negative sign is not in parentheses like $(-4)^2$.

$$\begin{aligned}-4^2&\neq(-4)^2\\-(4\cdot4)&\neq(-4)(-4)\\-16&\neq16\end{aligned}$$

Note: Problems 4.24–4.26 refer to the following expression: $2(5+3)^2$.

4.24 A student's incorrect attempt to simplify the expression appears below. Identify the error made by the student.

$$\begin{aligned} 2(5+3)^2 &= 2(5^2+3^2) \\ &= 2(25+9) \\ &= 2(34) \\ &= 68 \end{aligned}$$

The student incorrectly applies the distributive property, distributing the exponent through the terms in the parentheses. An exponent cannot be distributed to numbers within a quantity that are added or subtracted. Instead, the quantity $(5+3)$ should be simplified before the exponent is applied.

Note: Problems 4.24–4.26 refer to the following expression: $2(5+3)^2$.

4.25 Another student's incorrect attempt to simplify the expression appears below. Identify the error made by this student.

$$\begin{aligned} 2(5+3)^2 &= [2(5)+2(3)]^2 \\ &= [10+6]^2 \\ &= [16]^2 \\ &= 256 \end{aligned}$$

Like Problem 4.24, the student incorrectly applies the distributive property. You cannot multiply each of the terms within the parentheses by 2 because the quantity is raised to the second power. If the exponent were missing from the expression $2(5+3)$, it would be appropriate to apply the distributive property, producing the expression $2(5)+2(3)$. However, the presence of the exponent prevents you from applying the distributive property.

Note: Problems 4.24–4.26 refer to the following expression: $2(5+3)^2$.

4.26 Simplify the expression correctly.

If an expression contains grouping symbols (like parentheses) that define a quantity, that quantity must be simplified first. Calculate the sum $5+3$.

$$2(5+3)^2 = 2(8)^2$$

Expand the exponential expression and calculate the product.

$$\begin{aligned} 2(8)^2 &= 2(8\cdot 8) \\ &= 2(64) \\ &= 128 \end{aligned}$$

4.27 Simplify the expression: $(4 \cdot 3)^2$.

Begin by simplifying the expression within parentheses.

$$(4 \cdot 3)^2 = (12)^2$$

You're probably sensing a theme by now. Always simplify anything in parentheses, brackets, or other grouping symbols first.

Expand the exponential expression and calculate the product.

$$\begin{aligned}(12)^2 &= (12)(12) \\ &= 144\end{aligned}$$

4.28 Simplify the expression: $\left[2^3 \cdot 5\right]^2$.

The expression within brackets should be simplified before it is raised to the second power. Within the bracketed expression, you should expand and simplify 2^3 before multiplying by 5.

$$\begin{aligned}\left[2^3 \cdot 5\right]^2 &= [2 \cdot 2 \cdot 2 \cdot 5]^2 \\ &= [4 \cdot 2 \cdot 5]^2 \\ &= [8 \cdot 5]^2 \\ &= [40]^2\end{aligned}$$

Complete the problem by expanding the exponential expression and calculating the product.

$$\begin{aligned}[40]^2 &= 40 \cdot 40 \\ &= 1{,}600\end{aligned}$$

Order of Operations

Please excuse my dear Aunt Sally

4.29 Identify the order in which arithmetic operations should be completed.

One of the most popular mnemonic devices students use to memorize the order of operations is the phrase "Please Excuse My Dear Aunt Sally." The first letter of each word represents an operation, and the order of the words conveys the order in which the operations should be conducted:

Parentheses (or other grouping symbols) should be completed first. If an expression contains nested grouping symbols, you should simplify from the inside out, beginning with the innermost, nested quantity.

One set of grouping symbols is completely enclosed by another set of symbols, like the brackets are enclosed within parentheses here: $(1 + [2 + 3])$.

Exponents are the second priority in an expression.

Technically, addition and subtraction are the same operation—you're either adding a positive or a negative number, so that's why they're grouped in the same step. Same goes for multiplication and division. As you'll learn later in the book, division is just multiplication by a reciprocal.

Multiplication and **D**ivision should be completed in the same step, working from left to right. Even though the mnemonic study aid contains the words "My Dear," in that order, multiplication does *not* always occur before division—always simplify these two operations from left to right.

Addition and **S**ubtraction are undertaken in the same step once all other operations have been completed. These two operations should also be conducted from left to right.

It is important to note that the final two categories in the order of operations each consist of two operations, and the order in which each pair of operations is prioritized depends entirely upon which is written first, as you read from left to right.

4.30 Simplify the expression: $1 + 3 \cdot 4$.

The three numbers in this expression are connected by two operations: addition and multiplication. According to the order of operations stated in Problem 4.29, multiplication should occur before addition. Therefore, you should calculate the product of the numbers joined by the multiplication sign and then add 1 to that product.

$$\begin{aligned} 1+3\cdot 4 &= 1+12 \\ &= 13 \end{aligned}$$

4.31 Simplify the expression: $5 \cdot 2^4$.

Inserting parentheses changes the answer: $(5 \cdot 2)^4 = 10^4 = 10{,}000$.

This expression features two arithmetic elements: an exponent of 4 and a multiplication symbol. The order of operations indicates that exponents should be simplified first. One question remains: What is actually raised to the fourth power? Because no parentheses are present, you should raise 2 to the fourth power.

Expand the exponential expression, calculate its value, and then multiply that value by 5.

$$\begin{aligned} 5\cdot 2^4 &= 5(2\cdot 2\cdot 2\cdot 2) \\ &= 5(4\cdot 2\cdot 2) \\ &= 5(8\cdot 2) \\ &= 5(16) \\ &= 80 \end{aligned}$$

4.32 Simplify the expression: $18 - 5 + 2 - (-1)$.

All four numbers in this expression are connected by addition or subtraction signs. The order of operations dictates that both addition and subtraction occur in the same step, from left to right. However, before you begin simplifying the expression, you should eliminate the adjacent negative signs between 2 and 1. (See Problem 3.23 to review the technique for eliminating adjacent signs.)

$$18 - 5 + 2 - (-1) = 18 - 5 + 2 + 1$$

Combine the numbers, beginning on the left and progressing to the right.

$$\begin{aligned} 18-5+2+1 &= 13+2+1 \\ &= 15+1 \\ &= 16 \end{aligned}$$

4.33 Simplify the expression: $12 \div 4 \cdot (-6)$.

The order of operations stipulates that multiplication and division—both present in this expression—should be completed from left to right. Note that grouping symbols (in the form of parentheses) are present, but no expression is contained within.

You're supposed to simplify parentheses first, but the only thing in these parentheses is −6. There's nothing in there you can simplify.

$$\begin{aligned} 12 \div 4 \cdot (-6) &= 3 \cdot (-6) \\ &= -18 \end{aligned}$$

4.34 Simplify the expression: $[-1]^{2(-7+9)}$.

The power of this exponential expression contains multiplication and addition. Even though multiplication precedes addition in the order of operations, the addition problem occurs within parentheses, so it should be simplified first.

$$\begin{aligned} [-1]^{2(-7+9)} &= [-1]^{2(2)} \\ &= [-1]^4 \end{aligned}$$

Rewrite the exponential expression as a product and multiply the expanded values.

$$\begin{aligned} &= [-1][-1][-1][-1] \\ &= 1[-1][-1] \\ &= -1[-1] \\ &= 1 \end{aligned}$$

As the above steps demonstrate, a negative number raised to an even power produces a positive result; a negative number raised to an odd power produces a negative result.

4.35 Simplify the expression: $6^2 \div (-4+13)$.

Begin by simplifying the expression in parentheses.

$$6^2 \div (-4+13) = 6^2 \div 9$$

Exponents precede division in the order of operations.

$$\begin{aligned} &= (6 \cdot 6) \div 9 \\ &= 36 \div 9 \\ &= 4 \end{aligned}$$

4.36 Simplify the expression: $(1-2^3)^2 + 9$.

Raising to the third power is also called "cubing": 2^3 is read "two cubed."

Begin with the expression inside parentheses: $1 - 2^3$. Before you can subtract, you must first raise 2 to the third power: $2 \cdot 2 \cdot 2 = 8$.

$$\begin{aligned} (1-2^3)^2 + 9 &= (1-8)^2 + 9 \\ &= (-7)^2 + 9 \end{aligned}$$

Calculate the square of –7 and then add 9.

$$\begin{aligned} &= 49 + 9 \\ &= 58 \end{aligned}$$

4.37 Simplify the expression: $\dfrac{4(-10)+5^2}{12-3\cdot 2-1}$.

This fraction consists of two expressions that should be simplified independently. You may begin with either the numerator or denominator (or simplify both simultaneously). The numerator is simplified first in the steps below. Note that exponential expressions should be simplified before multiplying.

$$\begin{aligned} \frac{4(-10)+5^2}{12-3\cdot 2-1} &= \frac{4(-10)+25}{12-3\cdot 2-1} \\ &= \frac{-40+25}{12-3\cdot 2-1} \\ &= \frac{-15}{12-3\cdot 2-1} \end{aligned}$$

Now simplify the denominator, multiplying before you subtract.

$$
\begin{aligned}
&= \frac{-15}{12-6-1}\\
&= \frac{-15}{6-1}\\
&= \frac{-15}{5}
\end{aligned}
$$

$-3 \cdot 2 = -6$

Rewrite the fraction as a division problem and calculate the quotient.

$$
\begin{aligned}
&= -15 \div 5\\
&= -3
\end{aligned}
$$

4.38 Simplify the expression: $(8-2\cdot 7)^2 - 11^2$.

Simplify the expression within parentheses; according to the order of operations, you should multiply –2 and 7 and then add the product to 8.

$$
\begin{aligned}
(8-2\cdot 7)^2 - 11^2 &= (8-14)^2 - 11^2\\
&= (-6)^2 - 11^2
\end{aligned}
$$

Simplify the exponential expressions and then calculate the difference.

$$
\begin{aligned}
&= 36 - 121\\
&= -85
\end{aligned}
$$

4.39 Simplify the expression: $4^3 \cdot (-1) + [-3+3\cdot 5]^2$.

This expression contains addition, subtraction, multiplication, grouping symbols, and exponents. Begin by simplifying the grouped, bracketed expression. Note that within that quantity, multiplication must occur first.

$$
\begin{aligned}
4^3 \cdot (-1) + [-3+3\cdot 5]^2 &= 4^3 \cdot (-1) + [-3+15]^2\\
&= 4^3 \cdot (-1) + [12]^2
\end{aligned}
$$

Evaluate the exponential expressions.

$$
\begin{aligned}
&= (4\cdot 4\cdot 4)\cdot(-1) + [12\cdot 12]\\
&= (16\cdot 4)\cdot(-1) + [144]\\
&= 64\cdot(-1) + 144
\end{aligned}
$$

To complete the problem, multiply 64 by –1 and then add 144.

$$
\begin{aligned}
&= -64 + 144\\
&= 80
\end{aligned}
$$

4.40 Simplify the expression: $\dfrac{|1-7|^2}{(1-7)^2}$.

The absolute value symbols in the numerator are grouping symbols like the parentheses in the denominator. Both sets of grouping symbols in this problem contain the same expression, 1 – 7, and should be simplified first.

$$\frac{|1-7|^2}{(1-7)^2} = \frac{|-6|^2}{(-6)^2}$$

You are not finished simplifying the absolute value expression in the numerator until you actually calculate the absolute value.

$$= \frac{6^2}{(-6)^2}$$

Squaring a number and its opposite produce the same positive value: $6^2 = (-6)^2 = 36$.

$$= \frac{36}{36}$$

Express the fraction as a division problem and calculate the quotient.

$$= 36 \div 36$$
$$= 1$$

Chapter 5

INEQUALITIES

Greater than, less than, and some "equal to" sprinkled in

Whereas mathematics is often characterized—and accurately so—as a study of precisely preserved equality, more advanced mathematical studies require the investigation of inequalities. Restricting values to intervals of the real numbers, rather than insisting upon specific values, has sparked innovation that led to discoveries including differential calculus. This chapter does not delve as theoretically into the study of inequalities, sufficing to provide a rudimentary introduction that focuses on notation and one-dimensional representations.

Inequalities involve the symbols $<, >, \leq,$ and $\geq$. This chapter will make sure you understand what those symbols mean and how to graph statements that contain those symbols on a number line. It will also explain what interval notation is, so you can make sense of things like $[-1,3]$ and $(-\infty, 0)$. The last section ties inequalities back to the skills you've developed since Chapter 1—simplifying expressions that contain inequality symbols.

Inequality Statements

What's bigger than what?

If 14 > 3, then 3 < 14 is also a true statement: "Three is less than fourteen."

5.1 Write the following sentence as an inequality statement.

Fourteen is greater than three.

The inequality symbol ">" represents "greater than," so the mathematical representation of the sentence is $14 > 3$. To ensure that you apply the correct symbol, think of the inequality symbol as the head of an arrow. In this problem, the arrow would point to the right, toward 3. In all inequality statements, the arrow should point toward the lesser value; in this problem, 3 is less than 14.

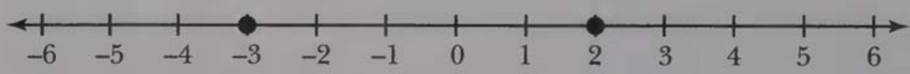
5.2 Write the following sentence as an inequality statement and explain why it is true.

Two is either greater than or equal to negative three.

Although inequality statements, by definition, are used to compare the sizes of unequal values, occasionally, the possibility of equality exists. This statement represents one such case: two may be greater than negative three or it may be equal to negative three. When the possibility of equality exists, use one of the following symbols: ≤ (less than or equal to) or ≥ (greater than or equal to).

$$2 \geq -3$$

Whereas +3 is greater than 2, –3 is less than 2. If you graph the values on a number line, the point representing –3 is left of the point representing 2.



The phrase "left of" is equivalent to the phrase "less than," when you are referring to a number line.

5.3 Write the following sentence as a simplified inequality statement.

Five is greater than the opposite of twelve minus four.

You could write the entire sentence as an inequality, but it is easier to simplify first. Specifically, the statement includes the phrase "the opposite of twelve minus four." Start by calculating the difference of 12 and 4.

$$12 - 4 = 8$$

The opposite of a number has the opposite sign. Because 8 is positive, its opposite is negative: –8. Therefore, the original sentence can be simplified to read, "Five is greater than negative eight." Use the inequality symbol > to represent "greater than."

$$5 > -(12-4)$$
$$5 > -8$$

5.4 Write the following sentence as a simplified inequality statement.

The product of two and negative nine is less than the absolute value of that product.

This statement compares the product of two and negative nine to the absolute value of that product. You are directed to simplify the statement, so you should begin by multiplying the numbers.

$$2(-9) = -18$$

Now calculate the absolute value of the product. Remember that the absolute value of a negative number is positive.

Flip back to Problem 3.8 if you need to review absolute values.

$$|-18| = 18$$

Now that you have identified the values described in the original sentence, you can express it mathematically. Use the symbol < to represent "less than."

$$2(-9) < |2(-9)|$$
$$-18 < 18$$

5.5 Write the following sentence as an inequality statement.

Nine more than the opposite of negative five is greater than or equal to the square of negative two.

This inequality statement compares two values: "nine more than the opposite of negative five" and "the square of negative two." Each value should be simplified independently.

"Nine more than the opposite of negative five" is written –(–5) + 9. You can simplify –(–5) one of two ways: either replace the adjacent signs "– –" with "+" or interpret –(–5) as "the opposite of –5." Either way, the result is +5.

This negative means "the opposite of"; the opposite of a number is the same thing as its negative. The opposite of a negative number is a positive number.

$$-(-5)+9 = 5+9$$
$$= 14$$

The second part of the sentence is "the square of negative two," or $(-2)^2$. Expand this expression and calculate its value.

$$\begin{aligned}(-2)^2 &= (-2)(-2)\\ &= 4\end{aligned}$$

Now that you have identified the two values that the inequality compares, you can express the inequality mathematically. Use the symbol "≥" to represent "greater than or equal to."

$$\begin{aligned}-(-5)+9 &\geq (-2)^2\\ 14 &\geq 4\end{aligned}$$

5.6 Write the following sentence as a compound inequality statement.

Ten is less than fifteen but greater than negative one.

A compound inequality statement combines two separate inequality statements into a single statement that contains two inequality signs. In this problem, you are given two separate inequality statements that involve the number 10. Express each as an inequality statement:

Ten is less than fifteen: $10 < 15$

Ten is greater than negative one: $10 > -1$

To construct a single, compound inequality statement, write the numbers –1, 10, and 15 in order, from least to greatest.

$$-1 \quad 10 \quad 15$$

If the original sentence had included "or equal to," you'd use ≤ instead of <.

There is no possibility of equality, so place "less than" signs between the values.

$$-1 < 10 < 15$$

You can read the compound statement from left to right as "–1 is less than 10, which is less than 15." Alternatively, you can write the values from largest to smallest and use "greater than signs."

$$15 > 10 > -1$$

While both techniques of expressing compound inequalities are correct, the former—using "less than symbols"—is a more common convention.

5.7 Write the following sentence as a compound inequality statement.

Negative one is greater than negative three but less than or equal to five.

This sentence compares –1 to two different values, –3 and 5:

Negative one is greater than negative three: $-1 > -3$

Negative one is less than or equal to five: $-1 \le 5$

Write the three values compared in the statement from least to greatest, as illustrated below.

$$-3 \quad -1 \quad 5$$

Insert "less than" symbols between the values, but note that a "less than or equal to" symbol should be used between –1 and 5, as the problem indicates the possibility of equality between those values.

$$-3 < -1 \le 5$$

5.8 Write the following sentence as a compound inequality statement.

Seven belongs to the interval bounded by six and eight, exclusive.

An interval is a portion of the number line defined by two values: a lower boundary and an upper boundary. This problem states that the number seven belongs to the portion of the number line that lies between six and eight. This is true, because six is less than seven and seven is less than eight.

You can express this relationship between the numbers as a compound inequality statement. List the values seven, six, and eight in order from least to greatest.

$$6 \quad 7 \quad 8$$

You use "less than" symbols to separate numbers that are written from least to greatest, but you need to determine whether to use "<" or "≤." Notice that the original statement concludes with the word "exclusive." That means that the boundaries six and eight are excluded from the interval—the portion of the number line that the statement describes begins at (but does not include) six and ends at (but does not include) eight.

If the boundaries of an interval are excluded, so is the possibility of equality. Therefore, you should use "<" rather than "≤" to separate the values in the compound interval.

You might be thinking, "Of course 7 doesn't equal 6 or 8! Why is anyone even wondering that?" The book is getting you used to the symbols, because later, when you use variables (like x and y) that represent unknown values, you won't be able to tell if things are equal just by looking at them.

$$6 < 7 < 8$$

5.9 Write the following sentence as a compound inequality statement.

Negative four belongs to the interval bounded by two and negative eleven, inclusive.

This problem states that –4 belongs to the portion of the number line bounded above by 2 and bounded below by –11. To express this statement as a compound inequality, write all three numbers in order, from least to greatest.

$$-11 \quad -4 \quad 2$$

You now need to determine whether to use "<" or "≤" symbols to complete the compound inequality. Notice that the original statement uses the word "inclusive" to describe the boundaries. Thus, the boundaries not only define the extreme values at the ends of the interval, they belong to the interval as well.

When the boundaries of an interval are included, you should write the accompanying compound inequality using "≤" symbols to indicate the possibility of equality.

$$-11 \le -4 \le 2$$

5.10 Write the following sentence as a compound inequality statement.

The interval containing zero has an excluded lower bound of negative five and an included upper bound of four.

This statement explains that 0 belongs to an interval with a lower boundary of –5 and an upper boundary of 4. Further, it explicitly states that the lower bound should be excluded (indicating that the symbol "<" should be used to separate –5 and 0 in the compound inequality) and the upper bound should be included (indicating that the symbol "≤" should be used to separate 0 and 4).

$$-5 < 0 \le 4$$

An interval may include one endpoint but exclude the other—both endpoints aren't automatically treated the same way.

Interval Notation

Section off the number line with brackets and parentheses

5.11 Identify the portion of the number line described below using interval notation.

All real numbers greater than one but less than eight

An interval is a portion of the real number line that is defined by two boundaries. Thus, intervals have many of the same characteristics as the compound inequality statements in Problems 5.6–5.10. In both cases, real numbers define the lower and upper boundaries of a range of real numbers, and in both cases, the segments of the number line explicitly include or exclude the boundary endpoints.

Interval notation consists of two numbers—the lower boundary of the interval (which is always written first) and the upper boundary of the interval. The numbers are separated by a comma and the pair of values is surrounded by brackets, parentheses, or a combination of one bracket and one parenthesis, depending upon whether or not the specific boundaries should be included on the interval.

This problem specifies a lower boundary of one and an upper boundary of eight, so write the numbers in order from least to greatest and separate them with a comma.

$$1,8$$

The problem specifically states that values in the interval must be greater than one and less than eight. Therefore, neither boundary is included in the interval. Use parentheses next to boundaries that are not included on an interval, as illustrated below.

$$(1,8)$$

The interval (1,8) represents the portion of the number line between (but not including) 1 and 8.

5.12 Identify the portion of the number line described below using interval notation.

All real numbers less than or equal to 19 and greater than or equal to –2

This problem lists the boundaries in a different order than Problem 5.11, but that does not affect the order in which the interval is written. The lower boundary should always precede the upper boundary in interval notation. Write the boundaries in order, from least to greatest, and separate them with a comma.

$$-2,19$$

Unlike Problem 5.11, both of the boundaries are included on the interval. Consider the lower boundary –2. According to the description of the interval, all values less than *or equal to* 19 are included and all values greater than *or equal to* –2 are included. Similarly, 19 also belongs to the interval.

Look for the words "or equal to." When you see them next to a number, that boundary is included, and you should use a bracket.

To indicate that the boundary of an interval should be included, place a bracket next to that boundary. In this problem, both boundaries are included, so brackets should be placed next to –2 and 19, as demonstrated below.

$$[-2,19]$$

Thus, the interval notation [–2,19] represents the set of real numbers on the number line between, and including, the boundaries of –2 and 19.

5.13 Identify the portion of the number line described below using interval notation.

"Reals" is a shorter way of saying "real numbers."

All reals greater than four but less than or equal to 20

Only one of the boundaries is included on this interval. In order to belong to this interval, a number must be greater than (and not equal to) 4 and less than (or equal to) 20. Hence, the lower boundary 4 is excluded but the upper boundary 20 is included. Place a parenthesis next to 4 and a bracket next to 20.

$$(4,20]$$

Numbers between 4 and 20 belong to the interval $(4,20]$, as does the number 20. However, 4 does not belong to the interval.

5.14 Identify the portion of the number line described below using interval notation.

All real numbers greater than three

Whereas Problems 5.11–5.13 featured intervals with two boundaries, this interval has only one explicitly stated boundary: three. More specifically, three is identified as the lower bound for the interval, because any number belonging to the interval must be greater than (and not equal to) three. No upper boundary is identified.

Correct interval notation absolutely requires two boundaries, but in this problem, any number greater than three qualifies for the interval. Consider a number line with the point three marked on it. Any number to the right of (and therefore greater than) three belongs to the interval—it extends infinitely to the right in the positive direction.

Because the interval has no upper boundary other than infinity itself, the symbol for infinity is used to represent the upper boundary: ∞. Write the boundaries next to each other and separate them with a comma. Note that ∞ should be written second, as it represents the upper boundary.

$$3,\infty$$

Always put a parenthesis next to ∞ for interval notation. Always, always, always.

To complete interval notation, you need to decide whether or not the endpoints should be included. The problem explicitly excludes the lower bound of three, so a parenthesis should be placed next to it. Because infinity is not a real number, you cannot include it in the interval, and a parenthesis should be used next to ∞ in interval notation.

$$(3,\infty)$$

5.15 Identify the portion of the number line described below using interval notation.

All real numbers less than or equal to negative five

Any number that is less than –5 (and the number –5, itself) belong to this interval. Like Problem 5.14, this interval has only one explicitly stated boundary—in this case, the upper bound. Note that Problem 5.14 uses the infinity symbol ∞ to represent an infinitely large upper boundary. Similarly, you use the negative infinity symbol $(-\infty)$ to represent an infinitely small (or negative) lower boundary.

List the lower bound of the interval followed by the upper bound, and separate them with a comma.

$$-\infty,-5$$

If the lower bound isn't explicitly stated, use $-\infty$. Use $+\infty$ (or just ∞) to represent a missing upper bound.

The problem indicates that –5 should be included in the interval (because it includes the modifier "or equal to"), so a bracket should be placed next to –5. An infinity symbol, whether positive or negative, cannot be included in any interval, so a parenthesis should be placed next to it, as illustrated below.

$$(-\infty,-5]$$

5.16 Identify a value that belongs to the following interval and construct a compound inequality statement that verifies your conclusion: [0,7].

The interval [0,7] consists of the real numbers between 0 and 7, including the boundaries themselves. There are infinitely many real numbers that belong to the interval, but one such example is the integer 5.

To construct a compound statement that claims 5 belongs to the interval, place the value between the lower and upper bounds of the interval, as illustrated below.

$$0 \quad 5 \quad 7$$

The "or equal to" phrases are required because of the brackets. Brackets mean equality is possible, so 0 and 7 also belong to the interval.

Like 5, the number you choose should be greater than (or equal to) 0 and less than (or equal to) 7. The brackets in the original interval direct you to place less than or equal to symbols between the values. (If the interval were written with parentheses instead, you would substitute less than symbols when constructing the compound inequality.) The final compound inequality is presented below.

$$0 \le 5 \le 7$$

5.17 Identify a value that belongs to the following interval and construct a compound inequality statement that verifies your conclusion: $[-9,1)$.

The lower bound of this interval is –9 and the upper bound is 1. Any real number between those values belongs to the interval, including—for example—the number –4. To construct a compound inequality statement, list the value you choose and the bounds of the interval in order, from least to greatest.

$$-9 \quad -4 \quad 1$$

To determine whether to place ≤ or < between the values, you should consult the original interval. A bracket is placed next to the lower bound –9, so it belongs to the interval and should be included with the ≤ symbol.

$$-9 \le -4 \quad 1$$

A parenthesis is placed next to the upper bound 1, indicating that it is not included in the interval. Exclude the upper boundary from the compound inequality statement using the < symbol.

$$-9 \le -4 < 1$$

5.18 Identify a value that belongs to the following interval and construct an inequality statement that verifies your conclusion: $(-\infty,11)$.

Any number between $-\infty$ and 11 belongs to this interval. More simply stated, you can choose any number less than (but not including) 11. For instance, the number 2 is less than 11, which you can state mathematically using the inequality below.

$$2 < 11$$

Unlike Problems 5.16–5.17, a compound inequality statement is not used.

That's because you're only given one real number (11). The $-\infty$ in the interval is a dead giveaway that you'll use the < sign, just like a $+\infty$ would indicate the > sign.

Graphing Inequalities

Darken sections of the number line

5.19 Graph the inequality on a number line: $x \ge 0$.

Although variables will not be discussed in detail until Chapter 15, they are necessary in the context of graphing problems. Specifically, the variable x is used to represent any number that satisfies the conditions set forth in the statement.

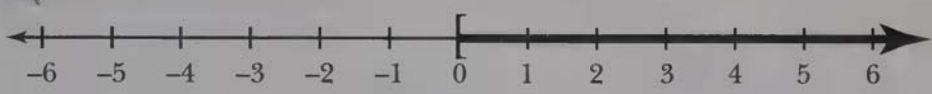
Given the inequality $x \ge 0$, your goal is to identify the portion of the number line that contains all of the numbers that are greater than or equal to 0. The mathematical expression $x \ge 0$ is preferred to the unwieldy sentence "any number greater than or equal to zero," which is why the variable x is necessary.

To graph the inequality, place a solid dot at 0 and draw an arrow that points to the right, in the same direction that the inequality sign would point if it were the head of an arrow. The arrow you draw should be thicker and darker than the number line so that you can distinguish between them, as illustrated below.

Any number along the darkened portion of the number line satisfies the inequality statement. For instance, $x = 1$, $x = 2$, $x = 3$, ... each belong to the darker, positive half of the number line, and each of those numbers satisfies the inequality $x \geq 0$. A solid dot is used on the graph to explicitly include 0 as part of the graph, as $0 \geq 0$ is a true statement.

Use closed dots when you see ≤ and ≥ symbols. Open dots are used for < and > symbols, as you'll see in Problems 5.22–5.23.

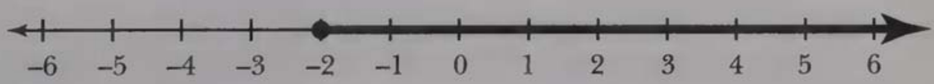
Some textbooks graph inequalities using parentheses and brackets, much like interval notation. The inequality expression $x \geq 0$ translates to the interval $[0,\infty)$. The graph below replaces the closed dot on the original graph with a bracket that indicates 0 should be included.



Graphs using interval notation are less conventional, so this book will default to inequality graphs that feature points, or "dots," instead.

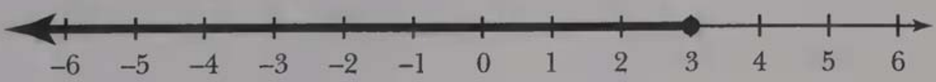
5.20 Graph the inequality on a number line: $x \geq -2$.

This inequality expression allows the possibility of equality, because ≥ includes the phrase "or equal to." Therefore, a closed dot should be used to mark −2 on the number line. Notice that the inequality symbol ≥ points to the right, so you should darken the portion of the number line to the right of −2, as illustrated below.



5.21 Graph the inequality on a number line: $x \leq 3$.

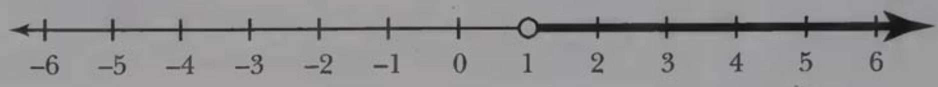
The inequality symbol ≤, like ≥, indicates that the adjacent number should be included in the graph. Therefore, a solid dot should be placed at 3 on the number line. Notice that the inequality symbol ≤ points to the left, so the portion of the number line that is left of 3 should be darkened, as illustrated below.



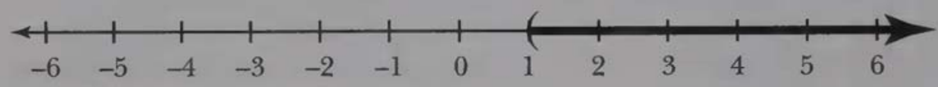
5.22 Graph the inequality on a number line: $x > 1$.

Unlike Problems 5.19–5.21, the boundary in this inequality expression should not be included, as the inequality symbol > does not allow for the possibility of equality. Recall that a solid dot is only used to mark an included boundary value on the number line. An open, or hollow, dot is used to mark an excluded boundary.

Place an open dot at 1 on the number line to represent the lower, but excluded, boundary of $x > 1$. Because the inequality sign points to the right, you should darken the portion of the number line to the right of 1.



Although it is less common, some textbooks use a parenthesis instead of an open dot on inequality graphs, as illustrated below.



5.23 Graph the inequality on a number line: $-5 > x$.

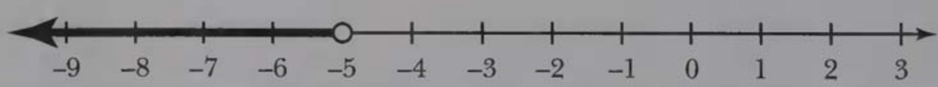
You darken the graph in the direction that the inequality symbol points. If you don't put x on the left side, you end up darkening the wrong side.

In order to use the graphing technique described in Problems 5.19–5.22, the variable x must appear on the left side of the inequality. However, when you reverse the sides of an inequality, you need to reverse the direction of the inequality symbol as well. In this problem, you should change the inequality symbol from > to <, as illustrated below.

$$x < -5$$

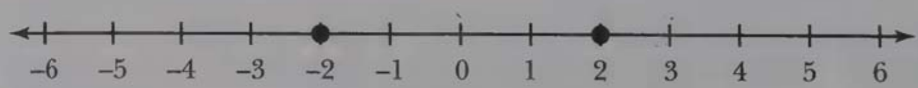
Reversing the inequality statement does not invalidate it. Rather, it represents an alternate way to state the same relationship: $-5 > x$ (negative five is greater than an unknown number x) and $x < -5$ (an unknown number is less than negative five) are logically equivalent statements.

To graph the inequality, place an open dot at –5. Draw a darkened arrow that points to the left, in the same direction as the inequality symbol in the statement $x < -5$.

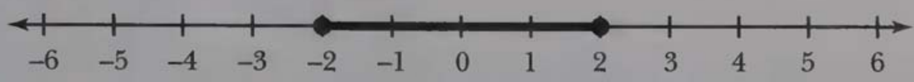


5.24 Graph the inequality on a number line: $-2 \le x \le 2$.

The graph of a compound inequality is a darkened portion of the number line that is bounded on both sides by a closed or an open dot. Both of the inequality symbols in this statement allow the possibility of equality, so the dots placed at –2 and 2 are both solid.



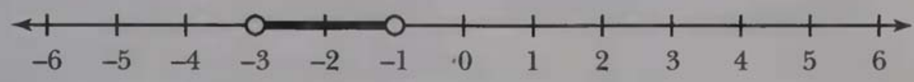
Complete the graph by darkening the number line between –2 and 2.



5.25 Graph the inequality on a number line: $-3 < x < -1$.

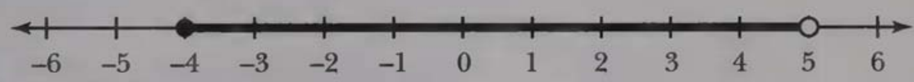
The boundaries of this compound inequality statement are –3 and –1. Both of the inequality symbols exclude the possibility of equality, so open dots should be used to mark the values on the number line. Darken the portion of the number line lying between the points.

No "or equal to" means hollow dots.



5.26 Graph the inequality on a number line: $-4 \le x < 5$.

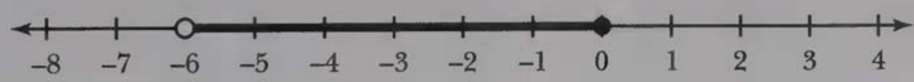
This compound inequality statement contains two different inequality symbols, $\le$ and $<$. Therefore, two different dots should be used to represent the boundaries of the interval. The symbol adjacent to –4 allows for equality, so place a solid dot at –4. The upper bound of the interval, however, is excluded, and should be marked with an open dot.



5.27 Graph the inequality on a number line: $0 \ge x > -6$.

This compound inequality is written differently than the inequalities in Problems 5.24–5.26, because the upper bound is written on the left and the lower bound is on the right. You can rewrite the statement by reversing the numbers and the inequality signs, but it is not necessary to complete the problem: $-6 < x \le 0$. Place an open dot at –6 and a closed dot at 0.

Unlike inequality graphs with one boundary (like Problems 5.19–5.23), you don't have to worry about reversing everything to get x in a certain place. If you do reverse everything, make sure the numbers and the inequality symbols move together.



Completing Inequality Statements

Simplify expressions and pick the right sign

5.28 Complete the inequality statement below, replacing the circle with one of the following symbols: <, >, or =.

$$4-3^2 \bigcirc 4+(-3)^2$$

Simplify the expressions on each side of the missing inequality symbol separately, following the order of operations described in Problem 4.29. To simplify the left expression, apply the exponent and then calculate the difference.

$$\begin{aligned} 4-3^2 &\bigcirc 4+(-3)^2 \\ 4-(3\cdot 3) &\bigcirc 4+(-3)^2 \\ 4-9 &\bigcirc 4+(-3)^2 \\ -5 &\bigcirc 4+(-3)^2 \end{aligned}$$

Simplify the right side of the statement, following the same order of operations.

$$\begin{aligned} -5 &\bigcirc 4+(-3)(-3) \\ -5 &\bigcirc 4+9 \\ -5 &\bigcirc 13 \end{aligned}$$

If you marked −5 and 13 on the number line, −5 would be further to the left. That means it's less than 13.

Negative numbers are less than positive numbers, so you should complete the statement using the < symbol.

$$-5 < 13$$

5.29 Complete the inequality statement below, replacing the circle with one of the following symbols: <, >, or =.

$$(8+1)\cdot 6 \bigcirc 8+1\cdot 6$$

Expressions within grouping symbols should always be completed first, so calculate the sum 8 + 1 before you multiply by 6.

$$\begin{aligned} (8+1)\cdot 6 &\bigcirc 8+1\cdot 6 \\ 9\cdot 6 &\bigcirc 8+1\cdot 6 \\ 54 &\bigcirc 8+1\cdot 6 \end{aligned}$$

Because no parentheses are present on the right side of the statement, multiplication precedes addition.

$$\begin{aligned} 54 &\bigcirc 8+6 \\ 54 &\bigcirc 14 \end{aligned}$$

The left side of the expression is greater than the right side, so complete the statement using the > symbol.

$$54 > 14$$

5.30 Complete the inequality statement below, replacing the circle with one of the following symbols: <, >, or =.

$$2|-1-4| \bigcirc 2|1+4|$$

Absolute value symbols are classified as grouping symbols, so the expressions within should be simplified before you multiply. On the left side of the expression, you should combine –1 and –4, take the absolute value, and then multiply the result by 2. Recall that the absolute value of a negative number is positive.

You don't have to simplify the sides one at a time like the book does—it's just going slow so you don't get lost. If you think you can simplify them both at the same time, go for it.

$$\begin{aligned} 2|-1-4| &\bigcirc 2|1+4| \\ 2|-5| &\bigcirc 2|1+4| \\ 2(5) &\bigcirc 2|1+4| \\ 10 &\bigcirc 2|1+4| \end{aligned}$$

Now simplify the right side of the expression. Recall that the absolute value of a positive number is positive.

$$\begin{aligned} 10 &\bigcirc 2|5| \\ 10 &\bigcirc 2(5) \\ 10 &\bigcirc 10 \end{aligned}$$

Both sides of the statement have the same value, so complete the statement using the = symbol.

$$10 = 10$$

5.31 Complete the inequality statement below, replacing the circle with one of the following symbols: <, >, or =.

$$\frac{-3+9\div 3}{2\cdot 6} \bigcirc \frac{2^4}{4^2}$$

Simplify the numerator and denominator of the left fraction independently. Make sure to divide 9 by 3 before subtracting 3 in the numerator.

$$\frac{-3+9\div 3}{2\cdot 6} \bigcirc \frac{2^4}{4^2}$$
$$\frac{-3+3}{2\cdot 6} \bigcirc \frac{2^4}{4^2}$$
$$\frac{0}{2\cdot 6} \bigcirc \frac{2^4}{4^2}$$
$$\frac{0}{12} \bigcirc \frac{2^4}{4^2}$$
$$0 \bigcirc \frac{2^4}{4^2}$$

$\frac{0}{12} = 0 \div 12$. In fact, 0 divided by any non-zero number is equal to 0.

Now simplify the right side of the expression.

$$0 \bigcirc \frac{2\cdot 2\cdot 2\cdot 2}{4\cdot 4}$$
$$0 \bigcirc \frac{4\cdot 2\cdot 2}{16}$$
$$0 \bigcirc \frac{8\cdot 2}{16}$$
$$0 \bigcirc \frac{16}{16}$$
$$0 \bigcirc 1$$

Any number divided by itself equals 1.

Zero is less than one, so complete the statement using the < symbol.

$$0 < 1$$

5.32 Complete the inequality statement below, replacing the circle with one of the following symbols: <, >, or =.

$$|3-2\cdot 7|^2 \bigcirc (3-2\cdot 7)^2$$

To simplify the left side of the statement, begin with the expression within absolute value symbols. Make sure to multiply 2 and 7 first and then subtract the product from 3. Once a single value lies within the absolute value symbols, calculate the absolute value and square it.

$$|3-2\cdot 7|^2 \bigcirc (3-2\cdot 7)^2$$
$$|3-14|^2 \bigcirc (3-2\cdot 7)^2$$
$$|-11|^2 \bigcirc (3-2\cdot 7)^2$$
$$11^2 \bigcirc (3-2\cdot 7)^2$$
$$121 \bigcirc (3-2\cdot 7)^2$$

The right side of the expression, aside from the absence of absolute value symbols, requires the same steps.

$$\begin{aligned} 121 &\bigcirc (3-14)^2 \\ 121 &\bigcirc (-11)^2 \\ 121 &\bigcirc 121 \end{aligned}$$

Both grouping symbols contain expressions equal to –11. On the left side, absolute values change that number to +11 before squaring it. On the right side of the expression; $(-11)^2$ also equals 121. Whether you square a negative or a positive number, the result is positive. Complete the statement using the = symbol.

$$121 = 121$$

From here until the end of the chapter, the book stops simplifying the sides in separate steps.

5.33 Complete the inequality statement below, replacing the circle with one of the following symbols: <, >, or =.

$$(-1)^7 \bigcirc (-1)^6$$

Expand the exponential expressions on both sides of the statement and multiply numbers in pairs, working from left to right, on each side. In each of the steps below, the underlined pair of numbers is multiplied to progress to the following step.

$$\begin{gathered} (-1)^7 \bigcirc (-1)^6 \\ \underline{(-1)(-1)}(-1)(-1)(-1)(-1)(-1) \bigcirc \underline{(-1)(-1)}(-1)(-1)(-1)(-1) \\ \underline{1(-1)}(-1)(-1)(-1)(-1) \bigcirc \underline{1(-1)}(-1)(-1)(-1) \\ \underline{(-1)(-1)}(-1)(-1)(-1) \bigcirc \underline{(-1)(-1)}(-1)(-1) \\ \underline{1(-1)}(-1)(-1) \bigcirc \underline{1(-1)}(-1) \\ \underline{(-1)(-1)}(-1) \bigcirc \underline{(-1)(-1)} \\ \underline{1(-1)} \bigcirc 1 \\ -1 < 1 \end{gathered}$$

5.34 Complete the inequality statement below, replacing the circle with one of the following symbols: <, >, or =.

$$(2^3)^3 \bigcirc (3^2)^2$$

This statement contains exponents raised to other exponents. According to the order of operations, you should simplify the expression within parentheses first.

There's an algebra property that says raising a power to a power is the same as multiplying the powers. In other words, $(2^3)^3 = 2^{3\cdot3} = 2^9$. This book won't get quite that advanced, though.

$$\left(2^3\right)^3 \circ \left(3^2\right)^2$$
$$(2\cdot 2\cdot 2)^3 \circ (3\cdot 3)^2$$
$$(4\cdot 2)^3 \circ (9)^2$$
$$8^3 \circ 9^2$$
$$8\cdot 8\cdot 8 \circ 9\cdot 9$$
$$64\cdot 8 \circ 81$$
$$512 > 81$$

5.35 Complete the inequality statement below, replacing the circle with one of the following symbols: <, >, or =.

$$\left[5(3-1\cdot 4)\right]^2 \circ 5(3-1\cdot 4)^2$$

Both sides of this expression contain similar elements, but the left side of the expression squares 5 and the parenthetical quantity, whereas the right side squares only the quantity. The other major dissimilarity between the sides is the pair of nested grouping symbols on the left side of the statement. Make sure to simplify the innermost quantity $(3 - 1 \cdot 4)$ first.

$$\left[5(3-1\cdot 4)\right]^2 \circ 5(3-1\cdot 4)^2$$
$$\left[5(3-4)\right]^2 \circ 5(3-4)^2$$
$$\left[5(-1)\right]^2 \circ 5(-1)^2$$

On the left side of the statement, calculate the product $5(-1)$ before squaring, as it appears within grouping symbols. However, on the right side of the statement, the order of operations directs you to simplify the exponential expression before multiplying.

$$[-5]^2 \circ 5[(-1)(-1)]$$
$$[-5][-5] \circ 5[1]$$
$$25 > 5$$

5.36 Complete the inequality statement below, replacing the circle with one of the following symbols: <, >, or =.

$$(8-5\cdot 2)^{6-3} \circ \left[13+(-3)^2-(-3)^2\right]^2$$

Simplify the entire expression within brackets on the right side of this statement before raising the value of that expression to the second power.

I know you're supposed to add from left to right, but +9 and −9 are right next to each other. It would save time to combine them and get 0, leaving 13 + 0 = 13.

$$(8-5\cdot 2)^{6-3} \circ \left[13+(-3)^2-3^2\right]^2$$
$$(8-10)^{6-3} \circ [13+(9)-(9)]^2$$
$$(-2)^3 \circ [22-9]^2$$
$$(-2)^3 \circ [13]^2$$

Simplify the exponential expressions to complete the problem.

$$(-2)(-2)(-2) \bigcirc [13][13]$$
$$4(-2) \bigcirc 169$$
$$-8 < 169$$

5.37 Complete the inequality statement below, replacing the circle with one of the following symbols: <, >, or =.

$$-4(5-3\cdot 2) \bigcirc (40 \div [-6+7\cdot 2])$$

Calculate the product of –3 and 2 to begin simplifying the left side of the statement. The right side contains nested quantities, so start by calculating the product $7 \cdot 2$ within the innermost quantity.

$$-4(5-3\cdot 2) \bigcirc (40 \div [-6+7\cdot 2])$$
$$-4(5-6) \bigcirc (40 \div [-6+14])$$
$$-4(-1) \bigcirc (40 \div 8)$$
$$4 < 5$$

5.38 Complete the inequality statement below, replacing the circle with one of the following symbols: <, >, or =.

$$16 \div 2 - 7\cdot 4 \bigcirc \left[-6^2 \div (-6+9\cdot 2)\right]^3$$

On the left side of this statement, the order of operations is division, multiplication, and then subtraction. The right side is a bit more complicated; it requires you to simplify the nested quantity, evaluate an exponent, divide, and then raise the result to the third power.

$$16 \div 2 - 7\cdot 4 \bigcirc \left[-6^2 \div (-6+9\cdot 2)\right]^3$$
$$8 - 7\cdot 4 \bigcirc \left[-6^2 \div (-6+18)\right]^3$$
$$8 - 28 \bigcirc \left[-6^2 \div 12\right]^3$$

Note that $-6^2 = -36$, which is read "the opposite of 6 squared equals –36." If the expression were written $(-6)^2$, "the square of negative six," then the answer would be +36.

$$-20 \bigcirc [-36 \div 12]^3$$
$$-20 \bigcirc [-3]^3$$
$$-20 \bigcirc [-3][-3][-3]$$
$$-20 \bigcirc 9[-3]$$
$$-20 > -27$$

5.39 Complete the inequality statement below, replacing the circle with one of the following symbols: <, >, or =.

$$9-30\div 6-[8+4\cdot 5]\bigcirc 2\cdot 7-32\div 4^2+11$$

Begin by simplifying the bracketed quantity on the left side of the statement and the exponential expression on the right side.

$$\begin{aligned} 9-30\div 6-[8+4\cdot 5] &\bigcirc 2\cdot 7-32\div 4^2+11 \\ 9-30\div 6-[8+20] &\bigcirc 2\cdot 7-32\div 16+11 \\ 9-30\div 6-28 &\bigcirc 14-32\div 16+11 \end{aligned}$$

Division is the next priority on both sides of the statement.

$$\begin{aligned} 9-5-28 &\bigcirc 14-2+11 \\ 4-28 &\bigcirc 12+11 \\ -24 &< 23 \end{aligned}$$

5.40 Complete the inequality statement below, replacing the circle with one of the following symbols: <, >, or =.

$$-6\left[28\div\left(4^2+[-12+10]\right)\right]\bigcirc\left[(10-4\cdot 6)+\left(9\div[12\div 4]\right)\right]+1$$

In order to proceed from one step to the next in the solution below, you need to simplify the underlined portion of the preceding step.

$$\begin{aligned} -6\left[28\div\left(4^2+\underline{[-12+10]}\right)\right] &\bigcirc \left[(10-4\cdot 6)+\left(9\div\underline{[12\div 4]}\right)\right]+1 \\ -6\left[28\div\left(\underline{4^2}-2\right)\right] &\bigcirc \left[(10-\underline{4\cdot 6})+(9\div 3)\right]+1 \\ -6\left[28\div\underline{(16-2)}\right] &\bigcirc \left[\underline{(10-24)}+(9\div 3)\right]+1 \\ -6[\underline{28\div 14}] &\bigcirc \left[-14+\underline{(9\div 3)}\right]+1 \\ \underline{-6[2]} &\bigcirc \underline{[-14+3]}+1 \\ -12 &\bigcirc \underline{-11+1} \\ -12 &< -10 \end{aligned}$$

The problem is pretty long, so the book helps you by underlining what needs to be done in each step.

Chapter 6

FACTORS, MULTIPLES, AND DIVISIBILITY

What divides into a number?

This chapter explores a key tenet of algebra, appropriately titled the Fundamental Theorem of Arithmetic, which states that every integer greater than or equal to two can be expressed as a unique product of prime factors. However complex this principle may appear at the outset of the chapter, once you have worked through these problems, you will understand the vocabulary and implications of this simple but important theorem.

Here's a quick preview of what you'll learn in this chapter, using the number 12 as an example:

- $2 \cdot 6 = 12$, so 2 and 6 are factors of 12
- 12 is a multiple of 2 and a multiple of 6
- 12 is divisible by 2 and 6
- 2 is prime, but 12 is a composite number
- The prime factorization of 12 is $2^2 \cdot 3$
- The least common multiple of 6 and 12 is 12
- The greatest common factor of 6 and 12 is 6

The first three bullets are pretty easy concepts, and the last four are a bit more complicated, but all of the topics in this chapter are related—they all have to do with numbers that divide into one another.

Basic Factoring Terminology

What do the terms in the chapter title mean?

6.1 Identify all factors of 20.

An integer is a number like −3 or 7 that has no decimal or fractional part.

A number is a factor of 20 if dividing 20 by that number produces a remainder of zero. Similarly, if a number is a factor of 20, you conclude that 20 is "divisible by" that number. All integers are divisible by 1, so technically 1 is a factor of all integers. Similarly, all integers are divisible by themselves; in this case, 20 is a factor of 20. Because all numbers contain these two factors by default, some texts consider them inconsequential and don't include them in lists of factors. In the interest of completeness, they are included in this book.

Beginning with 1, divide 20 by consecutive integers and identify the quotients that produce no remainder. The boxed numbers in the list below are factors of 20.

$$20 \div \boxed{1} = 20 \text{ r } 0$$
$$20 \div \boxed{2} = 10 \text{ r } 0$$
$$20 \div 3 = 6 \text{ r } 2$$
$$20 \div \boxed{4} = 5 \text{ r } 0$$
$$20 \div \boxed{5} = 4 \text{ r } 0$$
$$20 \div 6 = 3 \text{ r } 2$$
$$20 \div 7 = 2 \text{ r } 6$$
$$20 \div 8 = 2 \text{ r } 4$$
$$20 \div 9 = 2 \text{ r } 2$$
$$20 \div \boxed{10} = 2 \text{ r } 0$$
$$20 \div \boxed{20} = 1 \text{ r } 0$$

The quotients that correspond with the boxed factors in the list above are also factors of 20. For instance, the second row of the list states $20 \div 2 = 10$ r 0. Therefore, 2 is a factor of 10. However, when you divide 20 by 2, the result is 10, which is *also* a factor of 10, as evidenced by the last row of the list. Furthermore, when you divide 20 by 10, the result is 2.

Essentially, it is a duplication of effort to construct a list from 1 to 10, because all of the factors are identified in the first four rows: 1, 2, 4, 5, 10, and 20. Because 20 is divisible by 1, 2, 4, 5, 10, and 20, you can also conclude that 20 is a multiple of each number.

The list should stop with $20 \div 5$, because 5 was already identified as a factor in the previous row, which stated $20 \div 4 = 5$. As soon as you get a repeat factor, you can stop listing possible factors.

6.2 Identify all factors of 36.

Beginning with 2, divide 36 by consecutive integers until you have identified all of the factors. (Problem 6.1 states that every number is divisible by itself and 1, so 1 and 36 are factors of 36. No additional work is necessary to verify those factors.)

$$36 \div 2 = 18 \text{ r } 0$$

Dividing 36 by 2 produces a remainder of 0, so 2 and the resulting quotient 18 are both factors of 36. Continue dividing by consecutive integers greater than 2 until one of the factors repeats.

$$36 \div 3 = 12 \text{ r } 0$$
$$36 \div 4 = 9 \text{ r } 0$$
$$36 \div 5 = 7 \text{ r } 1$$
$$36 \div 6 = 6 \text{ r } 0$$

Dividing 36 by 6 produces no remainder, but the quotient (6) matches the number by which you divided. Once you encounter a repeated factor, the search for factors is complete. You conclude that the factors of 36 are 1, 2, 3, 4, 6, 9, 12, 18, and 36.

6.3 Based on your answer to Problem 6.2, determine whether 36 is a prime or composite number.

All positive integers fall into one of two categories: prime or composite. According to Problem 6.1, all numbers are divisible by themselves and divisible by 1. Prime numbers are divisible *only* by the numbers themselves and 1. Composite numbers, however, have more than those two factors. Problem 6.2 states that 36 has nine different factors including—but not limited to—1 and 36. Thus, 36 is a composite number.

6.4 Identify all factors of 25 and determine whether 25 is a prime or composite number.

Without performing any calculations, you can conclude that 25 has factors 1 and 25. To identify other factors, divide 25 by consecutive integers beginning with 2.

$$25 \div 2 = 12 \text{ r } 1$$
$$25 \div 3 = 8 \text{ r } 1$$
$$25 \div 4 = 6 \text{ r } 1$$
$$25 \div 5 = 5 \text{ r } 0$$

Dividing 25 by 5 produces a remainder of 0, so 5 is a factor of 25. Notice that the quotient (5) repeats the number by which you are dividing (5), so the search for factors is complete. The factors of 25 are 1, 5, and 25. Because 25 has more than the default factors of 1 and 25, it is a composite number.

6.5 Identify all factors of 13 and determine whether 13 is a prime or composite number.

Every number is divisible by 1 and the number itself. Thus, 1 and 13 are factors of 13. To identify additional factors, divide 13 by consecutive numbers beginning with 2.

$$13 \div 2 = 6 \text{ r } 1 \qquad 13 \div 8 = 1 \text{ r } 5$$
$$13 \div 3 = 4 \text{ r } 1 \qquad 13 \div 9 = 1 \text{ r } 4$$
$$13 \div 4 = 3 \text{ r } 1 \qquad 13 \div 10 = 1 \text{ r } 3$$
$$13 \div 5 = 2 \text{ r } 3 \qquad 13 \div 11 = 1 \text{ r } 2$$
$$13 \div 6 = 2 \text{ r } 1 \qquad 13 \div 12 = 1 \text{ r } 1$$
$$13 \div 7 = 1 \text{ r } 6$$

Stop at one less than the number, in this case 12. You already know 13 is a factor.

None of the quotients listed above have a remainder of 0, so none of the numbers between 2 and 12 are factors of 13. The only factors of 13 are the default factors of 1 and 13, so by definition, 13 is a prime number.

6.6 Identify all factors of 60 and determine whether 60 is a prime or composite number.

Divide 60 by consecutive numbers beginning with 2 until one of the factors repeats, as demonstrated below.

$$60 \div 2 = 30 \text{ r } 0 \qquad 60 \div 7 = 8 \text{ r } 4$$
$$60 \div 3 = 20 \text{ r } 0 \qquad 60 \div 8 = 7 \text{ r } 4$$
$$60 \div 4 = 15 \text{ r } 0 \qquad 60 \div 9 = 6 \text{ r } 6$$
$$60 \div 5 = 12 \text{ r } 0 \qquad 60 \div 10 = 6 \text{ r } 0$$
$$60 \div 6 = 10 \text{ r } 0$$

Dividing by 10 produces a quotient of 6. Notice that 6 and 10 were previously identified as factors of 60 ($60 \div 6 = 10$ r 0), so the search for factors is complete: 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, and 60.

6.7 Determine whether the following statement is true or false and explain your answer.

The product of two prime numbers is also a prime number.

This statement is false, and the easiest way to demonstrate why is with an example. Consider the prime numbers 5 and 7; calculate the product.

$$5 \cdot 7 = 35$$

This problem asks whether the product (35) of the prime numbers (5 and 7) is also prime. By definition, 35 is only prime if it has two factors: 1 and 35. However, the number 35 was generated by multiplying two numbers together, so 35 is divisible by those numbers.

$$35 \div 5 = 7 \quad \text{and} \quad 35 \div 7 = 5$$

If 35 is divisible by 5 and 7, then 5 and 7 are factors of 35, and 35 must be a composite number.

Divisibility Tests

Quick ways to identify factors of numbers

6.8 Apply a divisibility test to confirm that 7,158 is divisible by 2 and verify your answer.

Divisibility tests are quick, accurate ways to determine whether large numbers are divisible by smaller numbers without having to resort to trial and error. This is especially useful because long division is a cumbersome process when large numbers are involved, so long dividing only when you can be sure of a zero remainder is a sensible strategy.

The divisibility test for 2 is very straightforward: Any even number is divisible by 2. The last digit of 7,158 is 8, so 7,158 is an even number and, hence, divisible by 2. Verify the answer using long division.

A number is even if the ones digit is 0, 2, 4, 6, or 8.

$$\begin{array}{r}
3{,}579 \\
2\overline{)\ 7{,}158} \\
\underline{-6} \\
1\,1 \\
\underline{-1\,0} \\
15 \\
\underline{-14} \\
18 \\
\underline{-18} \\
0
\end{array}$$

The remainder of 7,158 ÷ 2 is 0, so 7,158 is divisible by 2.

Which also means:
(A) 2 is a factor of 7,158
(B) 7,158 is a multiple of 2

6.9 Apply a divisibility test to confirm that 1,365 is divisible by 3 and verify your answer.

A number is divisible by 3 if the sum of its digits is also divisible by 3. In other words, treat each digit of 1,365 as a separate number and calculate the sum.

$$1 + 3 + 6 + 5 = 15$$

The sum (15) is divisible by 3 because $15 \div 3 = 5$ has no remainder. You can conclude, therefore, that 1,365 is also divisible by 3. Verify your answer using long division.

$$\begin{array}{r} 455 \\ 3\overline{)\ 1{,}365} \\ \underline{-12} \\ 16 \\ \underline{-15} \\ 15 \\ \underline{-15} \\ 0 \end{array}$$

6.10 Apply a divisibility test to confirm that 3,788 is divisible by 4 and verify your answer.

If the last two digits of an integer are divisible by 4, then the entire number is divisible by 4 as well. In this problem, discard the thousands digit 3 and the hundreds digit 7; consider only the two-digit number 88 that remains. A quick long division problem verifies that 88 is, indeed, divisible by 4, and it is certainly preferable to dividing the entire number by 4.

Especially if 3,788 had a lot more digits.

$$\begin{array}{r} 22 \\ 4\overline{)\ 88} \\ \underline{-8} \\ 08 \\ \underline{-8} \\ 0 \end{array}$$

Because 88 is divisible by 4, you can conclude that 3,788 is also divisible by 4. Verify this using long division.

$$\begin{array}{r} 947 \\ 4\overline{)\ 3{,}788} \\ \underline{-36} \\ 18 \\ \underline{-16} \\ 28 \\ \underline{-28} \\ 0 \end{array}$$

6.11 Apply a divisibility test to confirm that 4,995 is divisible by 5 and verify your answer.

The divisibility test for 5 is very simple: a number is divisible by 5 if its ones digit is either 0 or 5. The ones digit of 4,995 meets this requirement, so 5 is a factor of 4,995. Check your answer by applying long division.

$$\begin{array}{r} 999 \\ 5\overline{)\ 4{,}995} \\ \underline{-4\ 5} \\ 49 \\ \underline{-45} \\ 45 \\ \underline{-45} \\ 0 \end{array}$$

6.12 Apply a divisibility test to confirm that 30,762 is divisible by 6 and verify your answer.

Because $2 \cdot 3 = 6$

A number is divisible by 6 only if it satisfies the divisibility tests for 2 and 3, described in Problems 6.8 and 6.9. In other words, the number must be even and the sum of its digits must be divisible by 3. The number 30,762 is even, so it is divisible by 2. Add the digits of 30,762 to determine whether the number is divisible by 3.

$$3 + 0 + 7 + 6 + 2 = 18$$

Because $18 \div 3 = 6$ r 0, you conclude that 18 is divisible by 3 and, by extension, 30,762 is divisible by 3 as well. Now that you have verified that 2 and 3 are both factors of 30,762, you know that 6 is a factor of 30,762 as well. Apply long division to check your answer.

$$\begin{array}{r} 5{,}127 \\ 6\overline{)\ 30{,}762} \\ \underline{-30} \\ 0\,7 \\ \underline{-\ 6} \\ 16 \\ \underline{-\ 12} \\ 42 \\ \underline{-\ 42} \\ 0 \end{array}$$

6.13 Apply a divisibility test to confirm that 357 is divisible by 7 and verify your answer.

In other words, multiply 7 by 2 to get 14 and then subtract 14 from 35.

The divisibility test for 7 differs a bit from the other tests explored in this section, but it is no more difficult. Begin by separating the ones digit of 357 from the rest of the number.

$$35 \quad 7$$

Double the ones digit that you detached in the previous step and subtract it from the other number.

$$\begin{array}{r} 35 \\ \underline{-14} \\ 21 \end{array}$$

Because the difference is divisible by 7 (the quotient $21 \div 7 = 3$ has no remainder), then the original number is divisible by 7 as well. Apply long division to verify your conclusion.

$$\begin{array}{r} 51 \\ 7\overline{)\ 357} \\ \underline{-35} \\ 07 \\ \underline{-7} \\ 0 \end{array}$$

You take the ones digit 4 off of 6,244 and double it to get 8.

6.14 Apply a divisibility test to confirm that 6,244 is divisible by 7 and verify your answer.

According to Problem 6.13, you should remove the ones digit of 6,244 and double it. This results in two separate numbers: 624 and 8. Subtract 8 from 624.

$$\begin{array}{r} 624 \\ \underline{-\ 8} \\ 616 \end{array}$$

Unlike Problem 6.13, it is not immediately obvious that the result is divisible by 7. However, the divisibility test for 7 can be applied multiple times. Separate 616 into two numbers by removing the ones digit (which results in 61 and 6). Double the digit you removed ($6 \cdot 2 = 12$) and subtract it from 61.

$$\begin{array}{r} 61 \\ \underline{-12} \\ 49 \end{array}$$

The difference is 49, which is more obviously divisible by 7 (especially when compared to the previous result of 616). Use long division to verify your answer.

$$\begin{array}{r} 892 \\ 7\overline{)\ 6{,}244} \\ \underline{-5\,6} \\ 64 \\ \underline{-63} \\ 14 \\ \underline{-14} \\ 0 \end{array}$$

6.15 Apply a divisibility test to confirm that 77,396,192 is divisible by 8.

The divisibility test for 8 is very similar to the divisibility test for 4, explained in Problem 6.10. In that procedure, you divide the last two digits of a number by 4 to determine whether the entire number is divisible by 4. In this problem, you divide the last *three* digits (the hundreds, tens, and ones places) by 8. If 8 is a factor of that three-digit number, then 8 is a factor of the original number as well.

Therefore, to determine whether 8 is a factor of 77,396,192, divide the last three digits (192) by 8.

$$\begin{array}{r} 24 \\ 8\overline{)\ 192} \\ \underline{-16} \\ 32 \\ \underline{-32} \\ 0 \end{array}$$

Because there is no remainder, 8 is a factor of 192 and, by extension, 77,396,192.

6.16 Apply a divisibility test to confirm that 51,093 is divisible by 9 and verify your answer.

The divisibility test for 3, explained in Problem 6.9, states that a number is divisible by 3 when the sum of its digits is divisible by 3. Similarly, a number is divisible by 9 when the sum of its digits is divisible by 9. Calculate the sum of the digits that comprise 51,093.

The "adding the digits" trick only works with 3 and 9.

$$5 + 1 + 0 + 9 + 3 = 18$$

Because $18 \div 9 = 2$ r 0, 18 is divisible by 9 and, by extension, 51,093 is divisible by 9. Apply long division to check your answer.

$$\begin{array}{r}
5{,}677 \\
9\overline{)\;51{,}093} \\
\underline{-45} \\
60 \\
\underline{-54} \\
69 \\
\underline{-63} \\
63 \\
\underline{-63} \\
0
\end{array}$$

6.17 Apply a divisibility test to confirm that 33,740 is divisible by 10 and verify your answer.

A number is divisible by 10 if its ones digit is 0. Therefore, 33,740 is divisible by 10. Check your answer using long division.

To divide 33,740 by 10, just erase the 0. The answer is 3,374. To multiply 33,740 by 10, place an extra 0 on the end of the number: 337,400.

$$\begin{array}{r}
3{,}374 \\
10\overline{)\;33{,}740} \\
\underline{-30} \\
37 \\
\underline{-30} \\
74 \\
\underline{-70} \\
40 \\
\underline{-40} \\
0
\end{array}$$

Factor Trees and Prime Factorization

Factor a number visually

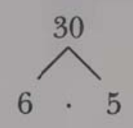
6.18 Describe the relationship between factor trees and the Fundamental Theorem of Arithmetic.

A factor tree is a tool that visually organizes the factors of a number. A number is first expressed as a product of two numbers, and branches are constructed from each of the composite factors until only prime factors remain. According to the Fundamental Theorem of Arithmetic, every positive integer can be expressed as a unique product of prime factors; the factor tree is a simple, straightforward way to identify those prime factors.

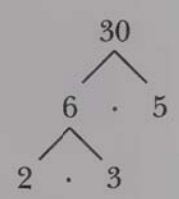
For example, the number 6 is equal to the product of prime numbers 2 and 3. By the way, 1 is not comsidered a prime number, so 2 is the smallest positive prime number.

6.19 Use a factor tree to identify the prime factorization of 30. Begin with the product $6 \cdot 5 = 30$.

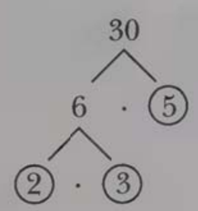
By directing you to use the product $6 \cdot 5$, this problem identifies the first two branches of the factor tree. Write 30, the number for which you are constructing the factor tree, and below it, list the pair of numbers 6 and 5, whose product is 30.



The factor 5 is a prime number, so the right branch of the factor tree is complete. However, the factor 6 is a composite number, so a new pair of branches needs to be constructed below it. Write 6 as a product of two numbers, but always avoid constructing branches using a factor of 1—rewriting 6 as the product $6 \cdot 1$ is not helpful. Rather, you should express 6 as the product $2 \cdot 3$, as illustrated below.



Each branch of the factor tree must end in a prime number. Consider the diagram below, which marks the end of each branch by circling the prime number at which it terminates.

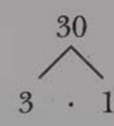


The prime factorization of 30 is the list of circled prime factors, multiplied by one another and written in order from least to greatest: $30 = 2 \cdot 3 \cdot 5$. To check your answer, ensure that each of the factors listed is prime and multiply them together to ensure the result is 30.

6.20 Verify your answer to Problem 6.19 by constructing a factor tree for 30 that begins with the product $3 \cdot 10$.

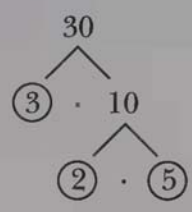
According to the Fundamental Theorem of Arithmetic, every positive integer has a unique prime factorization, regardless of the numbers that form the first two branches of the factor tree. Therefore, $3 \cdot 10$ should produce the same final prime factorization as the initial product $6 \cdot 5$ used in Problem 6.19.

Begin by writing 30 as the product $3 \cdot 10$, as directed by the problem.



So it doesn't matter if you start with 6 · 5 or 3 · 10 to construct the factor tree for 30—they both give you the same answer when you're done.

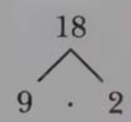
Notice that 3 is a prime number, so that branch of the factor tree is complete. However, 10 can be expressed as the product $2 \cdot 5$. Extend two new branches below 10 that end in the new factors, as illustrated below.



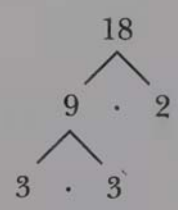
The end of each branch in the factor tree above is circled. Each of the circled numbers is prime, so the factor tree is complete. List the factors from least to greatest: $30 = 2 \cdot 3 \cdot 5$. This is the same prime factorization generated in Problem 6.19.

6.21 Identify the prime factorization of 18.

To construct the first branches of the factor tree, identify two numbers that have a product of 18. The solution below uses initial factors 2 and 9, but the product $3 \cdot 6$ will produce the same final answer.



The factor 2 is a prime number, but 9 can be expressed as the product $3 \cdot 3$.



Each of the branches now terminates in a prime number: the rightmost branch terminates at 2 and the two lower branches both terminate at 3. List the prime factors of 18 in order, from least to greatest.

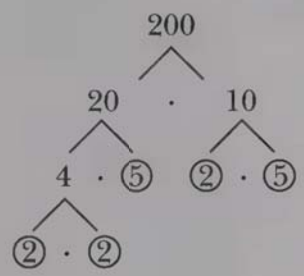
$$18 = 2 \cdot 3 \cdot 3$$

This prime factorization includes a repeated factor: 3. Textbooks conventionally list such factors using exponential notation, so express $3 \cdot 3$ as 3^2 in the final, prime factorization of 18.

$$18 = 2 \cdot 3^2$$

6.22 Identify the prime factorization of 200.

According to the divisibility test described in Problem 6.17, a number is divisible by 10 if its ones digit is 0. Divide 200 by 10 to identify a second factor: $200 \div 10 = 20$. Use the factors 10 and 20 to construct a factor tree.

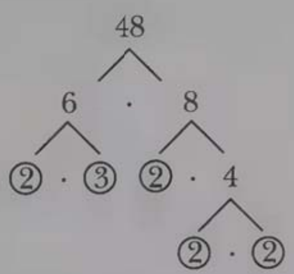


The prime factorization of 200 is $2^3 \cdot 5^2$.

6.23 Identify the prime factorization of 48.

According to the divisibility test discussed in Problem 6.12, the number 48 is divisible by 6 because it is even and the sum of its digits is divisible by 3. Divide 48 by 6 to identify another factor and use it in conjunction with 6 to construct a factor tree.

$48 \div 6 = 8$, so use 6 and 8 to start the factor tree.

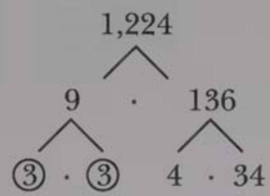


The prime factorization of 48 is $2^4 \cdot 3$.

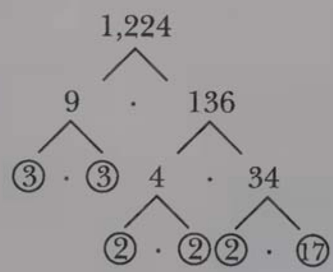
6.24 Identify the prime factorization of 1,224.

The sum of the digits of 1,224 ($1 + 2 + 2 + 4 = 9$) is divisible by 9, so according to the divisibility test described in Problem 6.16, 1,224 is divisible by 9 as well. Apply long division to determine that $1{,}224 \div 9 = 136$.

The last two digits of 136 are divisible by 4; according to the divisibility test described in Problem 6.10, 4 is also a factor of 136: $136 \div 4 = 34$.



To complete the factor tree, notice that 34 is divisible by 2 because it is an even number: $34 \div 2 = 17$.

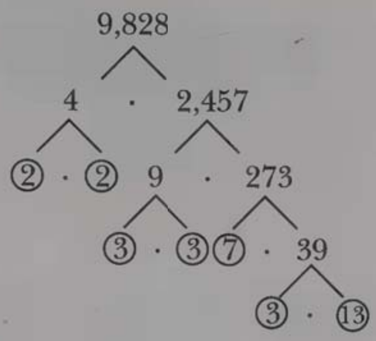


The prime factorization of 1,224 is $2^3 \cdot 3^2 \cdot 17$.

6.25 Identify the prime factorization of 9,828.

The last two digits of the number are divisible by 4, so 9,828 is divisible by 4 as well: $9{,}828 \div 4 = 2{,}457$. The digits of that quotient have a sum that is divisible by 9, so you can divide 2,457 by 9 to identify another factor: $2{,}457 \div 9 = 273$. According to the divisibility test described in Problems 6.13–6.14, 273 is divisible by 7: $27 - 6 = 21$, and $273 \div 7 = 39$. Apply all of these divisibility tests to construct the factor tree below.

Separate the last digit to create two numbers: 27 and 3. Double the smaller number and subtract it from the larger: $27 - 6 = 21$. That number is divisible by 7, so 273 is also divisible by 7.



The prime factorization of 9,828 is $2^2 \cdot 3^3 \cdot 7 \cdot 13$.

Least Common Multiple

Multiply to make numbers match

6.26 Explain why the least common multiple (LCM) of 8 and 24 is 24.

The least common multiple of a group of values is the smallest multiple those values have in common. To identify the least common multiple of 8 and 24, consider the larger of the two numbers. Because $8 \cdot 3 = 24$, by definition, 24 is a multiple of 8. Similarly, 24 is a multiple of itself—in fact, every number is a multiple of itself just as every number is a factor of itself—because $24 \cdot 1 = 24$. Thus, 24 is a multiple of both numbers.

24 is a MULTIPLE of 8 because you can MULTIPLY 8 by a whole number (3) to get 24.

However, 24 is not the only multiple of both numbers. To demonstrate this, double the larger of the two original numbers: $24 \cdot 2 = 48$. This new value is a multiple of 8 and 24: $8 \cdot 6 = 48$ and $24 \cdot 2 = 48$. Multiplying 24 by 3 produces another multiple of 8 and 24. Of the infinitely many common multiples of 8 and 24, the smallest (or least) common multiple is 24.

Note: Problems 6.27–6.28 demonstrate two different methods to calculate the least common multiple of 10 and 15.

6.27 Identify the least common multiple by listing multiples of the larger number.

This problem directs you to apply the technique described in Problem 6.26. Simply put, you should multiply the larger of the numbers (in this case 15) by consecutive integers beginning with 1.

$$1 \cdot 15 = 15$$

Note that the smaller number (10) does not divide evenly into 15. To continue searching for the least common multiple, now multiply 15 by 2.

And if that didn't work, you'd multiply 15 by 3, then by 4, then 5, until the smaller number divided in evenly.

$$2 \cdot 15 = 30$$

The smaller of the two original numbers (10) is a factor of 30, so 30 is the least common multiple of 10 and 15.

Note: Problems 6.27–6.28 demonstrate two different methods to calculate the least common multiple of 10 and 15.

6.28 Identify the least common multiple using prime factorizations.

Use the method described in Problems 6.19–6.25 to identify the prime factorizations of 10 and 15.

$$10 = 2 \cdot 5 \qquad 15 = 3 \cdot 5$$

You have identified three unique prime factors: 2, 3, and 5. To generate the least common multiple, multiply all three of the factors, raising each to its highest power from either factorization. In this case, 2, 3, and 5 are all raised to the first power, so the least common multiple is the product of 2, 3, and 5.

$$\begin{aligned} \text{LCM} &= 2 \cdot 3 \cdot 5 \\ &= 6 \cdot 5 \\ &= 30 \end{aligned}$$

This answer verifies the result of Problem 6.27.

The bigger number might be the LCM (like it was in Problem 6.26) so don't forget to try and divide the smaller number into the bigger number right away, before you multiply by 2, 3, 4, and so on.

Note: Problems 6.29–6.30 demonstrate two different methods to calculate the least common multiple of 12 and 16.

6.29 Identify the least common multiple by listing multiples of the larger number.

The larger of the two numbers in this problem is 16. Notice that $16 \cdot 1 = 16$ is not divisible by 12, so you should multiply 16 by 2.

$$16 \cdot 2 = 32$$

While 32 is (by definition) a multiple of 16, it is not a multiple of 12. Multiply 16 by the next largest whole number to continue the search for the least common multiple.

$$16 \cdot 3 = 48$$

Both 12 and 16 are factors of 48 ($48 \div 12 = 4$ and $48 \div 16 = 3$), so 48 is the least common multiple of 12 and 16.

Note: Problems 6.29–6.30 demonstrate two different methods to calculate the least common multiple of 12 and 16.

6.30 Identify the least common multiple using prime factorizations.

Calculate the prime factorizations of 12 and 16.

$$12 = 2^2 \cdot 3 \qquad 16 = 2^4$$

Two different factors appear in the prime factorizations: 2 and 3. Notice that the factor 2 is raised to different powers: 2^2 and 2^4. The least common multiple is the product of every unique factor raised to its highest power.

$$\begin{aligned} \text{LCM} &= 2^4 \cdot 3 \\ &= 16 \cdot 3 \\ &= 48 \end{aligned}$$

Use 2^4 instead of 2^2 to compute your LCM, because you're supposed to choose the highest power of each factor.

6.31 Identify the least common multiple of 28 and 72.

Construct the prime factorizations of both numbers.

$$28 = 2^2 \cdot 7 \qquad 72 = 2^3 \cdot 3^2$$

Three distinct prime factors are present; 2, 3, and 7. The least common multiple is the product of the factors when each is raised to its highest power. Specifically, the factor 2 should be raised to the third power rather than the second.

$$\begin{aligned} \text{LCM} &= 2^3 \cdot 3^2 \cdot 7 \\ &= 8 \cdot 9 \cdot 7 \\ &= 72 \cdot 7 \\ &= 504 \end{aligned}$$

6.32 Identify the least common multiple of 3, 9, and 15.

Construct the prime factorizations of 9 and 15; note that 3 is already a prime number.

$$3 \qquad 9 = 3^2 \qquad 15 = 3 \cdot 5$$

Two prime factors are present: 3 and 5. Raise each to its highest power and multiply the results.

$$\begin{aligned} \text{LCM} &= 3^2 \cdot 5 \\ &= 9 \cdot 5 \\ &= 45 \end{aligned}$$

6.33 Identify the least common multiple of 112, 600, and 882.

Construct the prime factorizations of all three numbers.

$$112 = 2^4 \cdot 7 \qquad 600 = 2^3 \cdot 3 \cdot 5^2 \qquad 882 = 2 \cdot 3^2 \cdot 7^2$$

There are four different prime factors that appear in one or more of the numbers: 2, 3, 5, and 7. Raise each to the highest power it appears in any of the factorizations and multiply the results to calculate the least common multiple.

$$\begin{aligned} LCM &= 2^4 \cdot 3^2 \cdot 5^2 \cdot 7^2 \\ &= 16 \cdot 9 \cdot 25 \cdot 49 \\ &= 176{,}400 \end{aligned}$$

Greatest Common Factor

The biggest thing that divides into a group evenly

6.34 Explain why the greatest common factor of 15 and 40 is 5.

The greatest common factor (GCF) of a set of values is the largest number by which each value in the set is divisible. Use the technique described in Problems 6.1–6.7 to list the factors of 15 and 40.

$$\begin{aligned} &15: 1, 3, \boxed{5}, 15 \\ &40: 1, 2, 4, \boxed{5}, 8, 10, 20, 40 \end{aligned}$$

The largest number that appears in both lists is 5, so 5 is the greatest common factor of 15 and 40.

Note: Problems 6.35–6.36 demonstrate two different methods to calculate the greatest common factor of 21 and 42.

6.35 Identify the greatest common factor by listing the factors of both numbers.

Apply the technique described in Problem 6.34, listing all factors of both numbers and then identifying the largest factor both lists have in common.

$$\begin{aligned} &21: 1, 3, 7, \boxed{21} \\ &42: 1, 2, 3, 6, 7, 14, \boxed{21}, 42 \end{aligned}$$

Both lists contain factors 1, 3, 7, and 21. The largest of these is the greatest common factor: 21.

Note: Problems 6.35–6.36 demonstrate two different methods to calculate the greatest common factor of 21 and 42.

6.36 Identify the greatest common factor using prime factorizations.

Construct the prime factorizations of each number using the method described in Problems 6.19–6.25.

$$21 = 3 \cdot 7 \qquad 42 = 2 \cdot 3 \cdot 7$$

There are three unique prime factors: 2, 3, and 7. Disregard any factors that are not common to both numbers. In this case, 2 is not a factor of 21, but both 3 and 7 are factors of 21 and 42. Therefore, the greatest common factor will be a product of 3 and 7. Raise each of the shared factors to the lowest power it is raised in either factorization. In this problem, all of the factors are raised to the first power, so the greatest common factor is the product of 3^1 and 7^1.

$$\begin{aligned} \text{GCF} &= 3 \cdot 7 \\ &= 21 \end{aligned}$$

This process is similar to the steps in Problem 6.28. To find the LCM, you take ALL of the factors and raise them to the HIGHEST power. To find the GCF, you take ONLY the common factors and raise them to the LOWEST power.

Though this technique differs from the technique in Problem 6.35, the result is the same.

Note: Problems 6.37–6.38 demonstrate two different methods to calculate the greatest common factor of 54 and 72.

6.37 Identify the greatest common factor by listing the factors of both numbers.

List all factors of both numbers.

$$54: 1, 2, 3, 6, 9, \boxed{18}, 27, 54$$
$$72: 1, 2, 3, 4, 6, 8, 9, 12, \boxed{18}, 24, 36, 72$$

The greatest common factor of 54 and 72 is 18. Notice that 72 has a lengthy list of factors. The larger the number, the longer the potential list of factors. Thus, the procedure described in Problems 6.36 and 6.38 is more practical for larger numbers.

Note: Problems 6.37–6.38 demonstrate two different methods to calculate the greatest common factor of 54 and 72.

6.38 Identify the greatest common factor using prime factorizations.

Construct the prime factorizations of both numbers.

$$54 = 2 \cdot 3^3$$
$$72 = 2^3 \cdot 3^2$$

Two prime factors are present (2 and 3) and the factors are common to both numbers. To calculate the greatest common factor, raise each common factor to its lowest power. In other words, 2^1 should be used in lieu of 2^3 and 3^2 should be used instead of 3^3.

$$\begin{aligned} \text{GCF} &= 2 \cdot 3^2 \\ &= 2 \cdot 9 \\ &= 18 \end{aligned}$$

6.39 Identify the greatest common factor of 60 and 108.

Construct the prime factorizations of the numbers.

$$60 = 2^2 \cdot 3 \cdot 5 \qquad 108 = 2^2 \cdot 3^3$$

There are two common prime factors: 2 and 3. Raise each to its lowest power and calculate the product.

$$\begin{aligned} \text{GCF} &= 2^2 \cdot 3 \\ &= 4 \cdot 3 \\ &= 12 \end{aligned}$$

6.40 Identify the greatest common factor of 40, 56, and 176.

Construct the prime factorizations of each number.

$$\begin{aligned} 40 &= 2^3 \cdot 5 \\ 56 &= 2^3 \cdot 7 \\ 176 &= 2^4 \cdot 11 \end{aligned}$$

The numbers 40, 56, and 176 have only one common prime factor: 2. The lowest power to which it is raised in the prime factorizations is 3, so the greatest common factor is $2^3 = 8$.

6.41 Identify the greatest common factor of 306, 459, and 561.

Identify the prime factorizations of each number.

$$\begin{aligned} 306 &= 2 \cdot 3^2 \cdot 17 \\ 459 &= 3^3 \cdot 17 \\ 561 &= 3 \cdot 11 \cdot 17 \end{aligned}$$

The numbers have two common prime factors: 3 and 17. The lowest power to which each is raised in the factorizations is 1, so the greatest common factor is $3 \cdot 17 = 51$.

561 has a prime factorization of $3^1 \cdot 11^1 \cdot 17^1$.

Chapter 7

INTRODUCTION TO FRACTIONS

One number divided by another number

Few mathematical concepts provoke visceral reactions in students as strongly as fractions, especially considering the time an average student spends studying fractions. However, an appropriate and methodical introduction to fractions helps to eliminate, or at least lessen, a too-typical histrionic knee-jerk response. This chapter will provide just such an introduction. Among the concepts that are investigated: proper and improper fractions, mixed numbers, equivalent fractions, and reducing fractions.

Lots of people are afraid of fractions, but when you break things down, fractions really aren't all that hard to understand. You're already used to performing arithmetic operations on single numbers. Now you'll be performing them on two numbers that are glued together. Ok, so it's a little trickier than that, but not much.

In Chapters 8 and 9, you learn how to add, subtract, multiply, and divide fractions, but this chapter only deals with the very basics. It answers questions like, "Which number is the denominator?" and "How do I reduce a fraction?".

Proper Fractions

Little number over big number

7.1 Identify the numerator and denominator of the fraction: $\frac{5}{6}$.

The fancy, official name for a fraction bar is "vinculum," but nobody (and I mean NOBODY) uses that term.

The number written above a fraction bar is the numerator, so the numerator of $\frac{5}{6}$ is 5. The denominator is the value below the fraction bar, so the denominator of $\frac{5}{6}$ is 6.

7.2 Given the fractions listed below, identify the proper fractions and explain your answer.

$$\frac{7}{1}, \frac{3}{2}, \frac{1}{3}, \frac{11}{4}, \frac{2}{5}, \frac{7}{6}, -\frac{3}{7}, \frac{6}{8}$$

A proper fraction has a numerator whose absolute value is less than the absolute value of its denominator. A fraction need not be reduced to be a proper fraction—the single qualification criterion is a denominator that exceeds the numerator. There are four proper fractions in the list: $\frac{1}{3}, \frac{2}{5}, -\frac{3}{7}$, and $\frac{6}{8}$.

7.3 Express the whole number 9 as a fraction.

Sometimes you'll see the word "rational number" instead of "fraction." Technically, any number that CAN BE written as a fraction is rational, so 9 is a whole number AND a rational number.

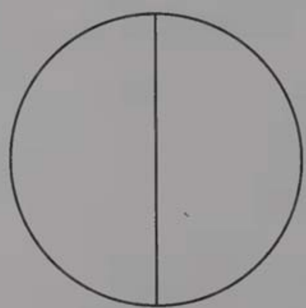
Any number divided by 1 is equal to the original number; in this case, $9 \div 1 = 9$. Express that quotient as a fraction.

$$9 \div 1 = \frac{9}{1}$$

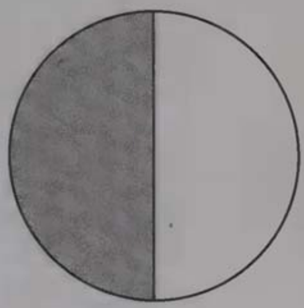
To express an integer as a fraction, use that integer as the numerator with a denominator of 1.

7.4 Draw a circle and shade $\frac{1}{2}$ of it.

A circle illustration is an effective way to visualize a simple fractional value. To construct one, draw a circle and divide it into equally sized regions according to the number in the denominator. The denominator of this problem is 2, so you should divide the circle into two equal parts, as illustrated below.

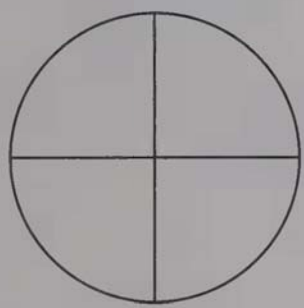


The value in the numerator of the fraction represents the number of sections that should be shaded. The fraction $\frac{1}{2}$ has a numerator of 1, so you should shade one of the two sections of the circle. It does not matter which section you shade and which is left unshaded—either way, half of the circle is shaded and half is not.

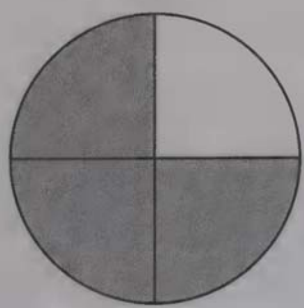


7.5 Draw a circle and shade $\frac{3}{4}$ of it.

Apply the procedure described in Problem 7.4. The denominator of the fraction is 4, so you should draw a circle that is divided into four equal parts, as illustrated below.



The numerator of the fraction is 3, so shade 3 of the 4 sections of the circle.

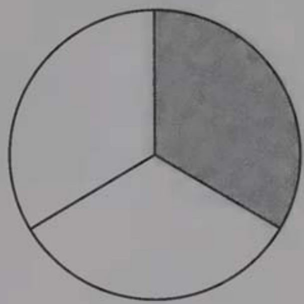


Three-fourths $\left(\frac{3}{4}\right)$ of the circle is shaded in the illustration above. Once again, it does not matter which three sections of the circle are shaded.

7.6 Draw a circle and shade $\frac{1}{3}$ of it.

Draw a circle and divide it into three equal regions, as illustrated below.

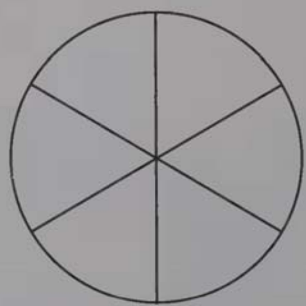
The numerator of $\frac{1}{3}$ is 1, so you should shade one of the sections.



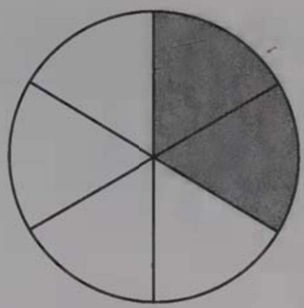
7.7 Draw a circle, shade $\frac{2}{6}$ of it, and verify that the shaded portion is equivalent to the shaded portion in Problem 7.6.

Divide a circle into six equal regions, as illustrated below.

Shade two regions that are next to each other. Pick the two sections that make up the larger, shaded section from Problem 7.6.



Shade two regions of the circle, as directed by the numerator. To help verify that $\frac{2}{6}$ and $\frac{1}{3}$ (the fraction from Problem 7.6) have the same value, it is helpful to shade two adjacent regions in the corresponding position.

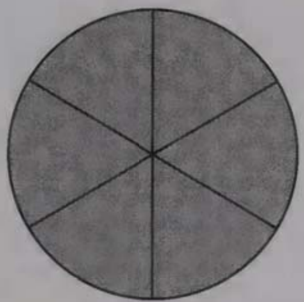


Two-sixths of a circle and one-third of a circle represent equivalent regions.

Problems 7.18–7.35 discuss equivalent fractions in more depth. Flip back here after you work through those problems, and the sentence "Reduce $\frac{2}{6}$ to get $\frac{1}{3}$" will make sense.

7.8 Draw a circle and shade $\frac{6}{6}$ of it. What conclusions can you draw based on your illustration?

Consider the illustration below, in which all six equal regions of a circle are shaded.



When you shade all six of six equal regions, you have shaded the entire circle. All six pieces combine to form one, complete whole. This conclusion translates into the following equation: $\frac{6}{6} = 1$. More generally, any fraction with equal values in the numerator and denominator is equal to one.

Mixed Numbers and Improper Fractions

Integers and fractions glued together and top-heavy fractions

7.9 Which of the two fractions below is an improper fraction? Explain your answer.

$$\frac{3}{8} \qquad \frac{8}{3}$$

The defining characteristic of an improper fraction is a numerator whose absolute value is larger than the absolute value of the denominator. Because $8 > 3$, the improper fraction is $\frac{8}{3}$.

In other words, when you ignore the signs, the top is bigger than the bottom.

7.10 When asked to provide an example of a mixed number, a student responds with the incorrect answer below. Identify two errors in the response and explain your answer.

$$4+\frac{3}{2}$$

A mixed number is the sum of a whole number and a fraction. However, the sum is understood and not explicitly indicated with an addition sign. Even though it is technically a sum, it is considered a single value, and an addition symbol would split a mixed number into two, separate values.

The second error in the student's response is the improper fraction $\frac{3}{2}$. A mixed number is defined as the sum of a whole number and a *proper* fraction. Therefore, the fraction portion of a mixed number must feature a numerator that is less than the denominator.

With a slight modification, you can use the numbers in the student's response to correctly construct the mixed number $4\frac{2}{3}$.

7.11 Express the improper fraction $\frac{5}{3}$ as a mixed number.

An improper fraction is precisely the opposite of a proper fraction. This allows you to divide the numerator by the denominator to produce a quotient that is greater than one. Apply long division to divide 5 by 3.

The absolute value of the numerator is greater than the absolute value of the denominator in an improper fraction.

$$\begin{array}{r} 1 \\ 3\overline{)\ 5} \\ \underline{-3} \\ 2 \end{array}$$

Thus, $5 \div 3 = 1$ r 2. Use the following steps to convert the results of long division into a mixed number:

1. The mixed number begins with a whole number equal to the quotient, 1 in this example.
2. The numerator of the fraction in the mixed number is the remainder, 2 in this example.
3. The denominator of the fraction in the mixed number is the denominator of the original, improper fraction, 3 in this example.

The improper fraction $\frac{5}{3}$ is equivalent to the mixed number $1\frac{2}{3}$.

7.12 Express the improper fraction $\frac{29}{4}$ as a mixed number.

Apply the method described in Problem 7.11: Divide 29 by 4 and make note of the resulting quotient and remainder.

$$\begin{array}{r} 7 \\ 4\overline{)\ 29} \\ \underline{-28} \\ 1 \end{array}$$

The quotient (7) is the whole part of the mixed number, the remainder (1) is the numerator of the proper fraction, and the original denominator (4) is the denominator. You conclude that the improper fraction $\frac{29}{4}$ is equivalent to the mixed number $7\frac{1}{4}$.

Here's where the parts of a mixed number come from: quotient $\frac{\text{remainder}}{\text{orig. denom.}}$

7.13 Express the improper fraction $\frac{86}{7}$ as a mixed number.

Identify the quotient and remainder of $86 \div 7$.

$$\begin{array}{r} 12 \\ 7\overline{)\ 86} \\ \underline{-7} \\ 16 \\ \underline{-14} \\ 2 \end{array}$$

The whole part of the mixed number is the quotient (12), the numerator is the remainder (2), and the denominator matches the denominator of the original, improper fraction (7). Thus, $\frac{86}{7} = 12\frac{2}{7}$.

7.14 Verify your answer to Problem 7.13 by transforming the final mixed number into an improper fraction.

The process outlined in Problem 7.11 allows you to express improper fractions as mixed numbers. A different technique is required to reverse the process.

To transform a mixed number into an improper fraction, multiply the denominator by the whole number, add the numerator, and keep the original denominator, as illustrated below:

$$\text{whole}\frac{\text{numerator}}{\text{denominator}} = \frac{(\text{denominator})(\text{whole}) + \text{numerator}}{\text{denominator}}$$

According to Problem 7.13, $\frac{86}{7}=12\frac{2}{7}$. To verify that the mixed number answer is correct, multiply its denominator by its whole number part and add the numerator.

$$\begin{aligned}7(12)+2&=84+2\\&=86\end{aligned}$$

The result is the numerator of the improper fraction. Its denominator matches the denominator of the mixed number, in this case 7. Therefore, $12\frac{2}{7}=\frac{86}{7}$. Because this improper fraction matches the improper fraction with which you started Problem 7.13, the mixed number representation is correct.

7.15 Express the mixed number $4\frac{1}{3}$ as an improper fraction.

Multiply the denominator (3) by the whole number (4) and add the numerator (1) to that product. This process identifies the numerator of the improper fraction.

$$\begin{aligned}3(4)+1&=12+1\\&=13\end{aligned}$$

The denominator of the improper fraction and the denominator in the mixed number are the same: 3. Therefore, $4\frac{1}{3}=\frac{13}{3}$.

7.16 Express the mixed number $2\frac{9}{13}$ as an improper fraction.

Multiply the denominator (13) by the whole number (2), and add the numerator (9).

$$\begin{aligned}13(2)+9&=26+9\\&=35\end{aligned}$$

Therefore, the numerator of the improper fraction is 35. The denominator is equal to 13, the denominator of the mixed number: $2\frac{9}{13}=\frac{35}{13}$.

7.17 Express the improper fraction $\frac{423}{11}$ as a mixed number.

Apply long division to identify the quotient and remainder of $423 \div 11$.

$$\begin{array}{r} 38 \\ 11\overline{)\,423} \\ \underline{-33} \\ 93 \\ \underline{-88} \\ 5 \end{array}$$

The whole portion of the mixed number is the quotient (38), the numerator of the proper fraction is the remainder (5), and the denominator is equal to the denominator of the improper fraction (11). Therefore, $\frac{423}{11} = 38\frac{5}{11}$.

7.18 Verify your answer to Problem 7.17 by converting the mixed number back into an improper fraction.

According to Problem 7.17, the improper fraction $\frac{423}{11}$ is equivalent to $38\frac{5}{11}$. Multiply the denominator (11) by the whole number (38) and add 5.

$$\begin{aligned} 11(38)+5 &= 418+5 \\ &= 423 \end{aligned}$$

The result is 423, the numerator of the improper fraction $\frac{423}{11}$, so the mixed number representation is correct.

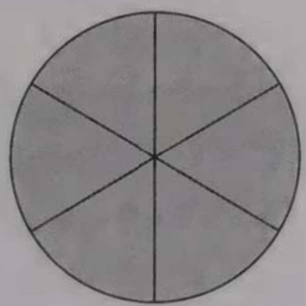
Equivalent Fractions

They feel the same but they don't look the same

7.19 Explain why multiplying the numerator and denominator of a fraction by the same number may change the numerator and denominator but does not change the value of the fraction.

Consider the diagram below, which also accompanies the answer to Problem 7.8. That problem instructs you to interpret the value of $\frac{6}{6}$, and it generates a visual representation of the fraction by shading all six sections of a circle that is divided into six equal regions.

A quick review on the rules of circle drawings for fractions: The bottom number of the fraction tells you how many sections the circle is divided into. The top number tells you how many of those sections to shade.



Problem 7.8 concludes that $\frac{6}{6}$ of a circle represents one, entire circle. Hence, $\frac{6}{6} = 1$. By extension, any fraction with the same, non-zero value in the numerator and denominator has a value of 1.

You're not allowed to divide by 0, so a fraction with 0 in the denominator breaks all kinds of rules.

According to the Identity Property of Multiplication, if you multiply a real number by 1, the result is the original real number—its *identity* is returned unaltered. For example: $3(1) = 3$ and $-6(1) = -6$.

This problem asks why multiplying a fraction's numerator and denominator by the same number does not change the value of that fraction. According to the logic outlined above, multiplying both parts of a fraction by the same number is akin to multiplying that fraction by 1, which does not alter the value of a fraction according to the Identity Property of Multiplication.

For example, multiplying the numerator and denominator of the fraction $\frac{4}{5}$ by 2 produces a fraction with different values but is nonetheless equivalent to $\frac{4}{5}$.

$$\frac{4\cdot 2}{5\cdot 2}=\frac{8}{10}$$

You are assured that $\frac{4}{5}$ and $\frac{8}{10}$ are equivalent fractions because multiplying by $\frac{2}{2}$ is the equivalent of multiplying by 1. Problems 7.27–7.34 explain in detail how to demonstrate the equivalency of the fractions by reducing numerators and denominators that share common factors.

In case you're curious, both of the fractions are equal to 0.5, but the book doesn't really start talking about decimals until Chapter 10.

7.20 Multiply the numerator and denominator of $\frac{1}{2}$ by 5 to generate an equivalent fraction.

According to Problem 7.19, multiplying the numerator and denominator of a fraction by the same integer generates an equivalent fraction, as the operations do not affect the value of the fraction.

$$\frac{1\cdot 5}{2\cdot 5}=\frac{5}{10}$$

Thus, $\frac{1}{2}$ and $\frac{5}{10}$ are equivalent fractions, two different representations of the same value.

7.21 Identify a fraction equivalent to $\frac{2}{3}$ that has a denominator of 21.

You are given the fraction $\frac{2}{3}$ and asked to identify the numerator of an equivalent fraction $\frac{?}{21}$. Recall that you can multiply the numerator and denominator of a fraction by the same number in order to generate an equivalent fraction, according to Problem 7.19. You already know that the denominator should be changed from 3 to 21, so divide the new denominator by the original denominator.

$$21 \div 3 = 7$$

When you multiply the original denominator (3) by 7, the result is the new denominator (21). If you also multiply the numerator of the original fraction by 7, you create an equivalent fraction with a denominator of 21, the precise fraction that is requested by this problem.

$$\frac{2\cdot 7}{3\cdot 7}=\frac{14}{21}$$

7.22 Identify a fraction equivalent to $\frac{1}{5}$ that has a denominator of 75.

You are asked to identify the numerator of a fraction with a denominator of 75. For the purposes of illustration, you can identify the fraction you are constructing as $\frac{?}{75}$. (Later in algebra, variables like *x* are used to represent unknown values, but a question mark will suffice for the moment.)

Divide the desired denominator by the original denominator.

$$\begin{array}{r} 15 \\ 5\overline{)\ 75} \\ \underline{-5} \\ 25 \\ \underline{-25} \\ 0 \end{array}$$

According to the quotient above, multiplying the original denominator (5) by 15 produces the desired denominator (75). Recall that a fraction's value is unchanged when both its numerator and denominator are multiplied by the same integer. Therefore, you can produce a fraction equivalent to $\frac{1}{5}$ that has a denominator of 75 by multiplying the numerator and denominator by 15.

$$\frac{1\cdot 15}{5\cdot 15}=\frac{15}{75}$$

7.23 Identify a fraction equivalent to $\frac{6}{7}$ that has a numerator of 48.

Divide the desired numerator by the original numerator.

$$48\div 6=8$$

Multiplying both the numerator and denominator of $\frac{6}{7}$ by 8 produces an equivalent fraction with a numerator of 48.

$$\frac{6\cdot 8}{7\cdot 8}=\frac{48}{56}$$

7.24 Identify an improper fraction equivalent to $6\frac{3}{4}$ that has a denominator of 36.

Before you can identify an equivalent improper fraction, apply the technique described in Problem 7.14 to convert the mixed number into an improper fraction.

$$\begin{aligned} 6\frac{3}{4} &= \frac{4 \cdot 6 + 3}{4} \\ &= \frac{24+3}{4} \\ &= \frac{27}{4} \end{aligned}$$

Given the improper fraction $\frac{27}{4}$, you are asked to identify an equivalent improper fraction with a denominator of 36. Divide the desired denominator (36) by the current denominator (4).

$$36 \div 4 = 9$$

To generate the equivalent improper fraction, multiply the numerator and denominator of $\frac{27}{4}$ by 9.

$$\frac{27 \cdot 9}{4 \cdot 9} = \frac{243}{36}$$

7.25 Identify an improper fraction equivalent to $7\frac{1}{9}$ that has a numerator of 192.

Express the mixed number as an improper fraction.

$$\begin{aligned} 7\frac{1}{9} &= \frac{9 \cdot 7 + 1}{9} \\ &= \frac{63+1}{9} \\ &= \frac{64}{9} \end{aligned}$$

Divide the desired numerator (192) by the numerator of the improper fraction (64).

$$192 \div 64 = 3$$

Multiplying the numerator and denominator of $\frac{64}{9}$ by 3 generates an equivalent fraction with a numerator of 192.

$$\frac{64 \cdot 3}{9 \cdot 3} = \frac{192}{27}$$

7.26 Identify a mixed number equivalent to $\frac{75}{4}$ that has a denominator of 156.

Although you are directed to report your final answer as a mixed number, leave the fraction in improper form until your calculations are complete. Your final step will be to convert the equivalent improper fraction into a mixed number.

Divide the desired denominator (156) by the given denominator (4).

$$\begin{array}{r} 39 \\ 4\overline{)\ 156} \\ \underline{-12} \\ 36 \\ \underline{-36} \\ 0 \end{array}$$

Multiply the numerator and denominator of $\frac{75}{4}$ by 39 to generate an equivalent improper fraction that has a denominator of 156.

$$\frac{75 \cdot 39}{4 \cdot 39} = \frac{2{,}925}{156}$$

$$\begin{array}{r} 75 \\ \underline{\times\ 39} \\ 675 \\ \underline{+2{,}250} \\ 2{,}925 \end{array}$$

To convert $\frac{2{,}925}{156}$ into a mixed number, divide 2,925 by 156.

$$\begin{array}{r} 18 \\ 156\overline{)\ 2{,}925} \\ \underline{-1\ 56} \\ 1{,}365 \\ \underline{-1{,}248} \\ 117 \end{array}$$

The quotient (18) is the whole portion of the mixed number, the remainder (117) is the numerator of the proper fraction, and the denominator of the equivalent fraction (156) matches the denominator of the mixed number.

$$\frac{2{,}925}{156} = 18\frac{117}{156}$$

Reducing Fractions

Transform fractions into a simpler, equivalent form

7.27 Reduce the fraction to lowest terms: $\frac{6}{10}$.

A fraction reduced to lowest terms is equivalent to the original fraction but has one specific, defining characteristic: The numerator and denominator cannot share any common factors. In other words, they cannot be divisible by the same number.

Just like you can multiply the top and bottom of a fraction by the same number to create an equivalent fraction, you can divide the top and bottom of a fraction by the same number without changing the fraction's overall value. Any number divided by 1 is the same number.

The fraction $\frac{6}{10}$ is not expressed in lowest terms because 6 and 10 share a common factor; they are both divisible by 2. Reduce the fraction by dividing both the numerator and denominator by the common factor.

$$\frac{6 \div 2}{10 \div 2} = \frac{3}{5}$$

The numerator and denominator no longer share any common factors, so the fraction is reduced to lowest terms. Note that $\frac{3}{5}$ and $\frac{6}{10}$ are equivalent fractions.

7.28 Reduce the fraction to lowest terms: $\frac{8}{16}$.

Like Problem 7.27, this fraction has a numerator and denominator that are even numbers and are thus divisible by 2. However, dividing by that common factor does not reduce the fraction to lowest terms, as demonstrated below.

$$\frac{8 \div 2}{16 \div 2} = \frac{4}{8}$$

You can once again divide by 2 on both sides of the fraction bar to continue the process of reducing to lowest terms, but the result will still not be expressed in lowest terms.

Consider the original fraction $\frac{8}{16}$ once again. Although the factor 2 is common to both the numerator and denominator, it is not the greatest common factor. Apply the technique described in Problem 6.34, listing the factors of 8 and 16, to identify the greatest common factor.

$$8: 1, 2, 4, \boxed{8}$$
$$16: 1, 2, 4, \boxed{8}, 16$$

To reduce the fraction $\frac{8}{16}$ to lowest terms in a single step, divide both the numerator and denominator by the greatest common factor, 8.

$$\frac{8 \div 8}{16 \div 8} = \frac{1}{2}$$

7.29 Reduce the fraction to lowest terms: $\frac{21}{49}$.

List the factors of 21 and 49 in order to identify the greatest common factor.

$$21: 1, 3, \boxed{7}, 21$$
$$49: 1, \boxed{7}, 49$$

Divide the numerator and denominator by 7, the greatest common factor of 21 and 49, to reduce the fraction to lowest terms.

$$\frac{21 \div 7}{49 \div 7} = \frac{3}{7}$$

7.30 Reduce the fraction to lowest terms: $\frac{15}{1{,}353}$.

The denominator of the fraction is a rather large number, especially when compared to the numerator. Rather than identify all factors of the denominator, list only the factors of 15.

One is a factor of every number, so you could argue that including it in the list is a waste of time.

$$15: 1, 3, 5, 15$$

Because 15 has only four factors (and only three nontrivial factors), you can test whether each, individually, is a factor of 1,353 as well. This expedites the process of identifying the greatest common factor in this example.

According to the divisibility test described in Problem 6.11, a number is divisible by 5 only if the digit in the ones place is 0 or 5. Because the ones digit of 1,353 is equal to neither of those numbers, 1,353 is not divisible by 5 and, by extension, not divisible by 15. The only remaining factors of 15 are 1 and 3.

If 1,353 were divisible by 15, it would also have to be divisible by the factors of 15 (including 5). You already know 1,353 can't be divided evenly by 5, so there's no way it could be divisible by 15.

Apply the divisibility test described in Problem 6.9: If the sum of the digits of 1,353 is divisible by 3, then 1,353 is divisible by 3 as well.

$$1 + 3 + 5 + 3 = 12$$

Because 12 is divisible by 3, 1,353 is also divisible by 3. Therefore, 3 is the greatest common factor of 15 and 1,353. In fact, it is the only common factor other than 1. Divide the numerator and denominator of the original fraction by 3 to reduce it to lowest terms.

$$\frac{15 \div 3}{1{,}353 \div 3} = \frac{5}{451}$$

7.31 Reduce the fraction to lowest terms: $\frac{27}{81}$.

Like Problem 7.30, this fraction contains one value that is significantly higher than the other. Rather than list all the factors of both to identify a greatest common factor, list the factors of the smaller number—in this case the numerator.

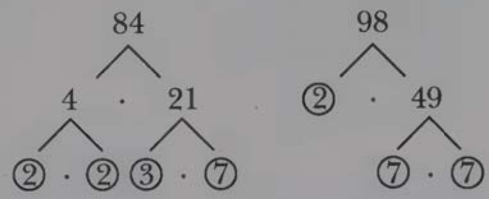
$$27: 1, 3, 9, 27$$

The denominator (81) is divisible by all four of the factors listed above. Thus, the largest factor of 27 is also the greatest common factor of 27 and 81. Divide the numerator and denominator by 27 to reduce the fraction to lowest terms.

$$\frac{27 \div 27}{81 \div 27} = \frac{1}{3}$$

7.32 Reduce the fraction to lowest terms and express it as a mixed number: $\frac{98}{84}$.

Identifying all factors of the numerator and denominator becomes more time consuming as the numbers in the fraction increase. Rather than list all factors of 98 and 84 to calculate the greatest common factor, apply the technique described in Problem 6.36: Construct factor trees to identify the prime factorizations of both numbers.



The quotient (1) is the whole number, the remainder (1) is the numerator, and the denominators (6) are equal.

The prime factorizations are $84 = 2^2 \cdot 3 \cdot 7$ and $98 = 2 \cdot 7^2$, so the numerator and denominator have two prime factors in common: 2 and 7. Multiply the lowest power of each to generate the greatest common factor.

$$\begin{aligned} GCF &= 2^1 \cdot 7^1 \\ &= 14 \end{aligned}$$

Divide the numerator and denominator of $\frac{98}{84}$ by 14 to reduce the fraction to lowest terms.

$$\frac{98 \div 14}{84 \div 14} = \frac{7}{6}$$

The problem directs you to report your answer as a mixed number. Note that $7 \div 6 = 1 \text{ r } 1$, so $\frac{7}{6} = 1\frac{1}{6}$.

7.33 Express the mixed number as an improper fraction reduced to lowest terms: $1\frac{60}{180}$.

Before you express the mixed number as an improper fraction, you should reduce the proper fraction portion to lowest terms. Both the numerator and denominator are divisible by 60.

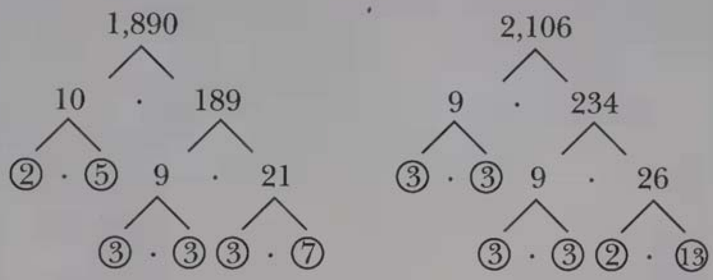
$$\frac{60 \div 60}{180 \div 60} = \frac{1}{3}$$

Therefore, $1\frac{60}{180} = 1\frac{1}{3}$. Express the mixed number as an improper fraction by multiplying the denominator (3) by the whole number (1) and adding the numerator (1). Divide the result by the denominator (3).

$$\begin{aligned}1\frac{1}{3} &= \frac{3\cdot 1+1}{3}\\ &= \frac{3+1}{3}\\ &= \frac{4}{3}\end{aligned}$$

7.34 Reduce the fraction to lowest terms: $\frac{1{,}890}{2{,}106}$.

The numerator and denominator contain large numbers, so you should construct factor trees to identify the greatest common factor of 1,890 and 2,106.



The prime factorizations are $1{,}890 = 2 \cdot 3^3 \cdot 5 \cdot 7$ and $2{,}106 = 2 \cdot 3^4 \cdot 13$, so the numbers share two prime factors: 2 and 3. Multiply the lowest power of each to generate the greatest common factor.

$$\begin{aligned}\text{GCF} &= 2\cdot 3^3\\ &= 2\cdot 27\\ &= 54\end{aligned}$$

Divide the numerator and denominator of the original fraction by 54 to reduce it to lowest terms.

$$\frac{1{,}890 \div 54}{2{,}106 \div 54} = \frac{35}{39}$$

Comparing Fractions

Which fraction is bigger?

The last seven problems of this chapter each contain a pair of fractions. Figure out which fraction in each pair is bigger (or show that they're equal).

Note: Complete the statements in Problems 7.35–7.41 by replacing the circle with one of the following symbols: <, >, or =. Explain your answers.

7.35 Compare the fractions: $\frac{3}{5} \bigcirc \frac{7}{5}$.

Given two fractions that share the same denominator, you should compare the numerators; the fraction with the larger numerator is also the greater fraction. This is not a stretch of logic, and is even easier to understand in the context of an example.

If you order pizzas and ask that each be cut into five equally sized slices, then each slice represents $\frac{1}{5}$ of a pizza. Therefore, $\frac{3}{5}$ of a pizza is equivalent to 3 slices and $\frac{7}{5}$ of a pizza is equivalent to 7 slices. Clearly, three slices are less than seven slices—in fact, $\frac{7}{5}$ of a pizza is more than one entire pizza and $\frac{3}{5}$ is less than one pizza. You conclude that $\frac{3}{5} < \frac{7}{5}$, because $3 < 7$.

Note: Complete the statements in Problems 7.35–7.41 by replacing the circle with one of the following symbols: <, >, or =. Explain your answers.

7.36 Compare the fractions: $\frac{19}{2} \bigcirc \frac{39}{4}$.

Unlike Problem 7.35, these fractions do not share the same denominator. Therefore, you cannot compare the numerators and draw a definitive conclusion about which is larger. However, the larger denominator (4) is a multiple of the smaller denominator (2). If you multiply the numerator and denominator of $\frac{19}{2}$ by 2, you generate an equivalent fraction that has a denominator of 4.

This is exactly what you're asked to do in Problems 7.21–7.26, so if you're working the problems in order, you've already practiced this skill.

$$\frac{19 \cdot 2}{2 \cdot 2} = \frac{38}{4}$$

Rewrite the original statement, replacing $\frac{19}{2}$ with the equivalent fraction $\frac{38}{4}$.

$$\frac{38}{4} \bigcirc \frac{39}{4}$$

Now that the fractions have a common denominator, it is clear that they are nearly equal. However, the numerator of the left fraction is less than the numerator of the right fraction ($38 < 39$), so $\frac{38}{4} < \frac{39}{4}$ and $\frac{19}{2} < \frac{39}{4}$. Remember that you can only compare numerators to identify the larger fraction when they share a common denominator.

Note: Complete the statements in Problems 7.35–7.41 by replacing the circle with one of the following symbols: <, >, or =. Explain your answers.

7.37 Compare the fractions: $\frac{2}{3} \bigcirc \frac{8}{15}$.

Multiply the numerator and denominator of $\frac{2}{3}$ by 5 to generate an equivalent fraction that has a denominator of 15.

$$\frac{2 \cdot 5}{3 \cdot 5} = \frac{10}{15}$$

Replace $\frac{2}{3}$ in the original statement with the equivalent fraction $\frac{10}{15}$.

$$\frac{10}{15} \bigcirc \frac{8}{15}$$

Given two fractions with the same denominator, the fraction with the larger numerator is also the larger fraction. Here, $10 > 8$, so the fraction on the left is larger.

$$\frac{10}{15} > \frac{8}{15} \text{ and } \frac{2}{3} > \frac{8}{15}$$

Note: Complete the statements in Problems 7.35–7.41 by replacing the circle with one of the following symbols: <, >, or =. Explain your answers.

7.38 Compare the fractions: $\frac{1}{9} \bigcirc \frac{1}{8}$.

Problem 7.35 describes a technique that identifies the larger of two fractions when they share the same denominator. This problem contains fractions that share a common *numerator*, and a different, but similar, technique exists for comparing these fractions.

If two or more positive fractions have the same numerator, the largest of the fractions will have the smallest denominator and vice versa. To understand this concept, once again think in context of slices of pizza. If you order one pizza that is divided into nine equal slices and another pizza that is divided into eight equal slices, then each slice of the first will be smaller than each slice of the second—the more pieces into which you divide a whole, the smaller those pieces are.

Same denominators: Bigger top number = bigger fraction

Same numerators: Smaller bottom number = bigger fraction

A fraction with denominator 9 is smaller than a fraction with denominator 8 (if both share the same numerator): $\frac{1}{9} < \frac{1}{8}$.

Note: Complete the statements in Problems 7.35–7.41 by replacing the circle with one of the following symbols: <, >, or =. Explain your answers.

7.39 Compare the fractions: $\frac{12}{7} \bigcirc \frac{6}{5}$.

There are multiple ways to correctly complete this problem, but whether you choose to rewrite the fractions with equal numerators or denominators, the symbol you choose as your final answer will be the same. The following solution is arguably the most expedient, because it only requires you to rewrite one of the fractions.

Note that 12 is a multiple of 6. Because $12 \div 6 = 2$, you can multiply the numerator and denominator of $\frac{6}{5}$ by 2 to generate an equivalent fraction that has a numerator of 12.

$$\frac{6\cdot 2}{5\cdot 2}=\frac{12}{10}$$

Rewrite the statement, replacing $\frac{6}{5}$ with the equivalent fraction $\frac{12}{10}$.

$$\frac{12}{7}\bigcirc\frac{12}{10}$$

According to Problem 7.38, given two fractions with the same numerator, compare the denominators: The smaller the denominator, the larger the fraction. Because $7 < 10$, you can conclude that $\frac{12}{7} > \frac{12}{10}$ and $\frac{12}{7} > \frac{6}{5}$.

Note: Complete the statements in Problems 7.35–7.41 by replacing the circle with one of the following symbols: <, >, or =. Explain your answers.

7.40 Compare the fractions: $3\frac{6}{7}\bigcirc\frac{80}{21}$.

And improper fractions are easier to deal with than mixed numbers.

It is easier to compare two fractions that are written in the same manner; rewrite $3\frac{6}{7}$ as an improper fraction.

$$\begin{aligned}3\frac{6}{7}&=\frac{7\cdot 3+6}{7}\\&=\frac{21+6}{7}\\&=\frac{27}{7}\end{aligned}$$

Rewrite the original statement, replacing the mixed number with the equivalent improper fraction.

$$\frac{27}{7}\bigcirc\frac{80}{21}$$

In order to compare the fractions, they need to share either the same numerator or the same denominator. Notice that 7 is a factor of 21; because $21 \div 7 = 3$, you can multiply the numerator and denominator of $\frac{27}{7}$ by 3 to generate an equivalent fraction that has a denominator of 21.

$$\frac{27\cdot 3}{7\cdot 3}=\frac{81}{21}$$

Rewrite the statement $\frac{27}{7}\bigcirc\frac{80}{21}$, replacing $\frac{27}{7}$ with the equivalent fraction calculated above.

$$\frac{81}{21}\bigcirc\frac{80}{21}$$

Given two fractions with the same denominator, the larger numerator identifies the larger fraction. In this problem, 81 > 80.

$$\frac{81}{21} > \frac{80}{21} \text{ and } 3\frac{6}{7} > \frac{80}{21}$$

Note: Complete the statements in Problems 7.35–7.41 by replacing the circle with one of the following symbols: <, >, or =. Explain your answers.

7.41 Compare the following fractions by expressing them in terms of a common denominator.

$$\frac{11}{18} \bigcirc \frac{8}{15}$$

While you can compare fractions that share a common numerator, it is much more useful to express fractions in terms of a common denominator. In order to express the fractions in this statement in terms of a common denominator, apply the technique described in Problems 6.26–6.33 to identify the least common multiple of 18 and 15.

As you learn pretty early in Chapter 8. Spoiler alert: you can't add or subtract fractions unless they have a common denominator.

Calculate the prime factorizations of the denominators.

$$18 = 2 \cdot 3^2 \qquad 15 = 3 \cdot 5$$

The factorizations contain three unique prime factors: 2, 3, and 5. To construct the least common multiple, calculate the product of those factors, raising each to the highest power it is raised in either factorization.

$$\begin{aligned} \text{LCM} &= 2 \cdot 3^2 \cdot 5 \\ &= 2 \cdot 9 \cdot 5 \\ &= 18 \cdot 5 \\ &= 90 \end{aligned}$$

Divide 90 by each of the denominators: 90 ÷ 18 = 5 and 90 ÷ 15 = 6. These results are the numbers by which you should multiply the numerator and denominator of each fraction in order to create equivalent fractions with common denominator 90. Specifically, 90 ÷ 18 = 5, so both parts of the fraction with denominator 18 should be multiplied by 5.

$$\frac{11 \cdot 5}{18 \cdot 5} = \frac{55}{90}$$

Similarly, 90 ÷ 15 = 6, so both parts of the fraction with denominator 15 should be multiplied by 6.

$$\frac{8 \cdot 6}{15 \cdot 6} = \frac{48}{90}$$

Rewrite the original statement, replacing the fractions with the equivalent fractions calculated above.

$$\frac{11}{18} \bigcirc \frac{8}{15}$$
$$\frac{55}{90} \bigcirc \frac{48}{90}$$

The fractions share a common denominator, so the fraction with the larger numerator is greater: $\frac{55}{90} > \frac{48}{90}$ and $\frac{11}{18} > \frac{8}{15}$.

Chapter 8

ADDING AND SUBTRACTING FRACTIONS

Make sure the denominators match

The actual process used to add and subtract fractions is extraordinarily easily. Essentially, you add or subtract fractions simply by adding or subtracting their numerators. However, this process can only begin once the fractions share a common denominator, and this is the most difficult part of the process.

By design, Chapters 6 and 7 have introduced all of the prerequisite skills you will need to understand and master the problems in this chapter, so make sure you have carefully worked through all of the examples in those chapters before undertaking the problems here.

Chapters 6 and 7 had their share of weird topics, including problems that asked you to generate equivalent fractions with a specific numerator or denominator. It's not terribly difficult to multiply the top and bottom of a fraction by the same number, but what's the point?

You may have wondered, "What is a least common multiple, and why has the book barely mentioned it after the end of Chapter 6? Why did I spend so much time learning something useless?" On second thought, I hope you're really not thinking like that, because it probably means you're frustrated, or grumpy, or maybe hungry, but I digress.

I have good news! If you invested time on the problems in the last two chapters, Chapter 8 is going to be a LOT easier. If you've never added or subtracted fractions before, you should know that it can be very confusing unless you understand what's going on. Don't worry, though. As long as you've done all the problems in Chapters 6 and 7, the problems in Chapter 8 will be very straightforward, and you'll breeze right through them.

Least Common Denominator

Works like least common multiple

8.1 List the five smallest common multiples of 6 and 9 by listing multiples of both numbers, and identify the least common multiple (LCM).

To generate multiples of a number, multiply it by consecutive positive integers beginning with 1. The first 15 multiples of 6 are listed below as an example.

$$\begin{array}{lll} 6\cdot 1=6 & 6\cdot 6=\boxed{36} & 6\cdot 11=66 \\ 6\cdot 2=12 & 6\cdot 7=42 & 6\cdot 12=\boxed{72} \\ 6\cdot 3=\boxed{18} & 6\cdot 8=48 & 6\cdot 13=78 \\ 6\cdot 4=24 & 6\cdot 9=\boxed{54} & 6\cdot 14=84 \\ 6\cdot 5=30 & 6\cdot 10=60 & 6\cdot 15=\boxed{90} \end{array}$$

Now multiply 9 by consecutive positive integers, beginning with 1, to generate the first 10 multiples of 9.

$$\begin{array}{ll} 9\cdot 1=9 & 9\cdot 6=\boxed{54} \\ 9\cdot 2=\boxed{18} & 9\cdot 7=63 \\ 9\cdot 3=27 & 9\cdot 8=\boxed{72} \\ 9\cdot 4=\boxed{36} & 9\cdot 9=81 \\ 9\cdot 5=45 & 9\cdot 10=\boxed{90} \end{array}$$

The five multiples common to both lists are 18, 36, 54, 72, and 90. Of these common multiples, 18 is the least: $18 < 36 < 54 < 72 < 90$. Therefore, the least common multiple of 6 and 9 is 18.

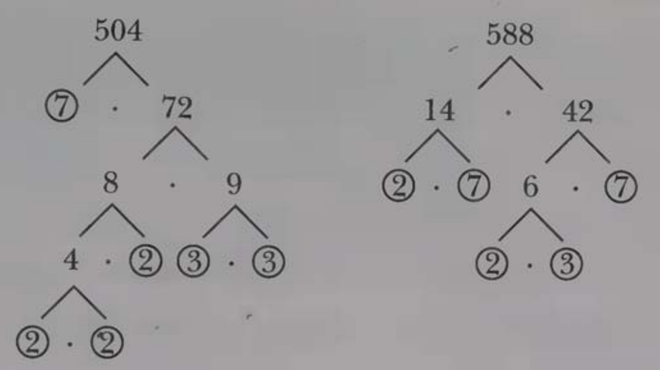
Notice that you had to list 15 multiples of 6 but only 10 multiples of 9 in order to identify the four required common multiples. Furthermore, it is not necessary to list the factors of both numbers to identify the least common multiple. Problem 6.27 outlines a slightly more efficient technique: Multiply the largest number by 2, 3, 4, 5, … until you identify the number into which everything divides evenly.

$18 \div 9 = 2$ r 0 and $18 \div 6 = 3$ r 0. Both numbers divide evenly into 18 because neither quotient has a remainder.

In this example, you would multiply 9 by 2 to get 18. Because both 6 and 9 divide evenly into 18, the process is complete and 18 is the least common multiple. This method is preferred to the brute force technique of listing all multiples of both numbers.

8.2 Identify the least common multiple of 504 and 588 using prime factorizations.

Problem 6.28 describes how to identify a least common multiple using prime factorizations. Begin by building factor trees for both numbers.



The prime factorizations are the products of the circled prime numbers that end each branch of the factor trees.

$$504 = 2^3 \cdot 3^2 \cdot 7$$
$$588 = 2^2 \cdot 3 \cdot 7^2$$

Choose 2^3 instead of 2^2, 3^2 instead of 3, and 7^2 instead of 7.

The prime factorizations contain three unique prime factors: 2, 3, and 7. To calculate the least common multiple, multiply the factors, raising each to the highest power from either factorization.

$$\begin{aligned} \text{LCM} &= 2^3 \cdot 3^2 \cdot 7^2 \\ &= (8)(9)(49) \\ &= (72)(49) \\ &= 3{,}528 \end{aligned}$$

The least common multiple of 504 and 588 is 3,528. Notice that both 504 and 588 divide into 3,528 evenly, as demonstrated below.

$$3{,}528 \div 504 = 7$$
$$3{,}528 \div 588 = 6$$

8.3 Explain how the concepts of least common multiple and least common denominator (LCD) are related, and describe how to choose between the methods presented in Problems 8.1 and 8.2 when you are required to calculate a least common multiple during a problem.

Given two or more fractions, you can identify the least common denominator by calculating the least common multiple of the denominators. For example, given two fractions with denominators of 6 and 9, according to Problem 8.1, the least common denominator is 18 because it is the least common multiple of 6 and 9.

Most people try the easy way (list multiples of the largest number until all the numbers divide evenly) first, but once you've multiplied the largest number by 6 or 7 and still haven't found the LCM, it's probably time to break out the factor trees.

The techniques presented in Problems 8.1 and 8.2 are both valid methods to accurately calculate a least common multiple (and hence a least common denominator), but the method in Problem 8.1 works best given small numbers or cases in which one of the denominators is clearly a multiple of the other. If presented with large denominators, or if your attempts to apply the method of Problem 8.1 prove unfruitful, you should opt for the prime factorization technique described in Problem 8.2.

8.4 Write the fractions in terms of the least common denominator: $\frac{1}{2}$ and $\frac{5}{8}$.

Consider the denominators of the fractions: 2 and 8. The larger denominator is also the least common denominator because 8 is evenly divisible by 8 and 2. To express $\frac{1}{2}$ in terms of the least common denominator, you need to generate a fraction equivalent to $\frac{1}{2}$ that has a denominator of 8. Divide the desired denominator (8) by the current denominator (2).

$$8 \div 2 = 4$$

The numerator and denominator of $\frac{1}{2}$ should be multiplied by 4, the quotient calculated above.

$$\frac{1 \cdot 4}{2 \cdot 4} = \frac{4}{8}$$

The LCD can't be smaller than the largest denominator.

The fractions $\frac{1}{2}$ and $\frac{5}{8}$, written in terms of the least common denominator, are $\frac{4}{8}$ and $\frac{5}{8}$.

8.5 Write the fractions in terms of the least common denominator: $\frac{11}{18}$ and $\frac{2}{3}$.

Like Problem 8.4, the larger denominator of these fractions is also the least common denominator; 3 and 18 both divide evenly into 18. Therefore, the fraction $\frac{11}{18}$ is, by default, already written in terms of the least common denominator. To construct a fraction equivalent to $\frac{2}{3}$ that has a denominator of 18, divide the desired denominator (18) by the current denominator (3).

$$18 \div 3 = 6$$

Multiply the numerator and denominator of $\frac{2}{3}$ by 6 to generate an equivalent fraction.

$$\frac{2 \cdot 6}{3 \cdot 6} = \frac{12}{18}$$

Therefore, the fractions written in terms of a common denominator are $\frac{11}{18}$ and $\frac{12}{18}$.

8.6 Write the fractions in terms of the least common denominator: $\frac{3}{4}$ and $\frac{5}{6}$.

Unlike Problems 8.4–8.5, the larger denominator of these two fractions is not the least common denominator, because 6 is not evenly divisible by 4, the smaller denominator. Apply the method described at the end of Problem 8.1: Multiplying the largest denominator by consecutive integers beginning with 2 until both denominators divide into the product evenly.

Begin by multiplying the largest denominator (6) by 2: $6 \cdot 2 = 12$. Notice that 12 is evenly divisible by both denominators: $12 \div 4 = 3$ and $12 \div 6 = 2$. Therefore, 12 is the least common denominator. Use the quotients that demonstrate even divisibility to identify the numbers by which to multiply the fractions: $\frac{3}{4}$ should by multiplied by $\frac{3}{3}$ and $\frac{5}{6}$ should be multiplied by $\frac{2}{2}$.

These are the numbers you just got when you divided 12 by 4 and 6.

$$\frac{3 \cdot 3}{4 \cdot 3} = \frac{9}{12} \qquad \frac{5 \cdot 2}{6 \cdot 2} = \frac{10}{12}$$

The fractions $\frac{3}{4}$ and $\frac{5}{6}$, written in terms of the least common denominator, are $\frac{9}{12}$ and $\frac{10}{12}$.

8.7 Write the fractions in terms of the least common denominator: $\frac{1}{4}$ and $\frac{3}{5}$.

Multiply the larger denominator (5) by consecutive integers, beginning with 2, until you produce a number into which both denominators (4 and 5) divide evenly.

$$5 \cdot 2 = 10$$
$$5 \cdot 3 = 15$$
$$5 \cdot 4 = 20$$

Notice that $20 \div 4 = 5$ r 0 and $20 \div 5 = 4$ r 0, so 20 is evenly divisible by 4 and 5. Thus, it is the least common denominator. Because dividing 20 by 4 results in 5, both parts of the fraction with a denominator of 4 should be multiplied by 5.

$$\frac{1 \cdot 5}{4 \cdot 5} = \frac{5}{20}$$

Because $20 \div 5 = 4$, both parts of the fraction with a denominator of 5 should be multiplied by 4.

$$\frac{3 \cdot 4}{5 \cdot 4} = \frac{12}{20}$$

The fractions $\frac{1}{4}$ and $\frac{3}{5}$, written in terms of the least common denominator, are $\frac{5}{20}$ and $\frac{12}{20}$.

8.8 Write the fractions in terms of the least common denominator: $\frac{4}{9}$ and $\frac{7}{12}$.

Multiply the larger denominator (12) by consecutive integers until the product is divisible by both denominators (9 and 12).

$$12 \cdot 2 = 24$$
$$12 \cdot 3 = 36$$

The least common denominator is 36. Because $36 \div 9 = 4$, multiply the numerator and denominator of $\frac{4}{9}$ by 4.

$$\frac{4 \cdot 4}{9 \cdot 4} = \frac{16}{36}$$

Similarly, $36 \div 12 = 3$, so the numerator and denominator of $\frac{7}{12}$ should be multiplied by 3.

$$\frac{7 \cdot 3}{12 \cdot 3} = \frac{21}{36}$$

The original fractions, written in terms of the least common denominator, are $\frac{16}{36}$ and $\frac{21}{36}$.

8.9 Write the fractions in terms of the least common denominator: $\frac{11}{12}$ and $\frac{13}{15}$.

List multiples of the larger denominator until you identify a value that is evenly divisible by both denominators.

$$15 \cdot 2 = 30$$
$$15 \cdot 3 = 45$$
$$15 \cdot 4 = 60$$

The least common denominator is 60. Because $60 \div 12 = 5$, the numerator and denominator of $\frac{11}{12}$ should be multiplied by 5; similarly, $60 \div 15 = 4$, so the numerator and denominator of $\frac{13}{15}$ should be multiplied by 4.

$$\frac{11 \cdot 5}{12 \cdot 5} = \frac{55}{60} \qquad \frac{13 \cdot 4}{15 \cdot 4} = \frac{52}{60}$$

The original fractions, written in terms of the least common denominator 60, are $\frac{55}{60}$ and $\frac{52}{60}$.

RULE OF THUMB

This is the last time the book explains how to go from LCD to equivalent fractions with that denominator. Once more for review: Divide the desired denominator by the current denominator and then multiply both parts of the fraction by the result.

8.10 Write the fractions in terms of the least common denominator: $\frac{1}{8}$ and $\frac{1}{10}$.

List multiples of the larger denominator until you identify a number into which both denominators divide evenly.

$$10\cdot 2=20$$
$$10\cdot 3=30$$
$$10\cdot 4=40$$

Express both fractions in terms of the least common denominator, 40.

$$\frac{1\cdot 5}{8\cdot 5}=\frac{5}{40}\qquad \frac{1\cdot 4}{10\cdot 4}=\frac{4}{40}$$

8.11 Write the fractions in terms of the least common denominator: $\frac{13}{14}$ and $\frac{2}{21}$.

Although these denominators contain larger numbers than most of the denominators of the preceding problems in this section, it is still appropriate to apply the technique described and modified in Problem 8.1. List multiples of 21 until the result is also divisible by 14.

Even though LCMs of larger denominators (like in Problems 8.12–8.13) are usually easier to build using factor trees and prime factorizations, don't forget to try the multiples technique first, just in case.

$$21\cdot 2=42$$

The first multiple on the list is evenly divisible by 14.

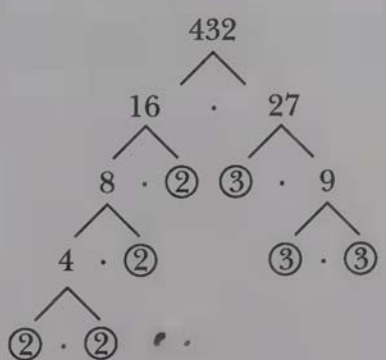
$$42\div 14=3$$

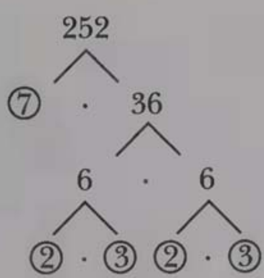
Write both fractions in terms of the least common denominator, 42.

$$\frac{13\cdot 3}{14\cdot 3}=\frac{39}{42}\qquad \frac{2\cdot 2}{21\cdot 2}=\frac{4}{42}$$

8.12 Write the fractions in terms of the least common denominator: $\frac{101}{432}$ and $\frac{97}{252}$.

The denominators of the fractions are large, three-digit numbers. Rather than list multiples of the larger denominator (432) until you identify a number into which the smaller denominator (252) also divides evenly, you should apply the method described in Problem 8.2. Begin by building factor trees for each denominator to identify their prime factorizations.





The prime factorizations of the denominators are $432 = 2^4 \cdot 3^3$ and $252 = 2^2 \cdot 3^2 \cdot 7$. Three unique prime factors are present: 2, 3, and 7. The least common denominator is the product of all three factors, when each is raised to its highest power from either factorization.

Remember: The factors don't have to appear in both factorizations, like when you calculate GCFs.

$$\begin{aligned} \text{LCD} &= 2^4 \cdot 3^3 \cdot 7 \\ &= 16 \cdot 27 \cdot 7 \\ &= 432 \cdot 7 \\ &= 3{,}024 \end{aligned}$$

Divide the least common denominator (3,024) by each of the original denominators.

$$3{,}024 \div 432 = 7 \qquad 3{,}024 \div 252 = 12$$

The quotients above are the numbers by which you should multiply the numerators and denominators of the original fractions in order to express them in terms of the least common denominator.

$$\frac{101 \cdot 7}{432 \cdot 7} = \frac{707}{3{,}024} \qquad \frac{97 \cdot 12}{252 \cdot 12} = \frac{1{,}164}{3{,}024}$$

8.13 Write the fractions in terms of the least common denominator: $\frac{1}{4}$, $\frac{2}{5}$, and $\frac{5}{6}$.

When problems contain more than two fractions for which you are required to calculate the least common denominator, it is more likely that you will need to apply the prime factorization method described in Problem 8.2. Each additional fraction vastly increases the probability that the multiples technique of Problem 8.1 will not be an efficient technique.

The denominators in this problem are small, so there is no need to build factorization trees to identify the prime factorizations.

5 is a prime number, so its prime factorization is 5, or 5^1 if you're feeling wacky.

$$\begin{aligned} 4 &= 2^2 \\ 5 &= 5 \\ 6 &= 2 \cdot 3 \end{aligned}$$

There are three unique prime factors: 2, 3, and 5. The least common denominator is the product of all three when each is raised to its highest power.

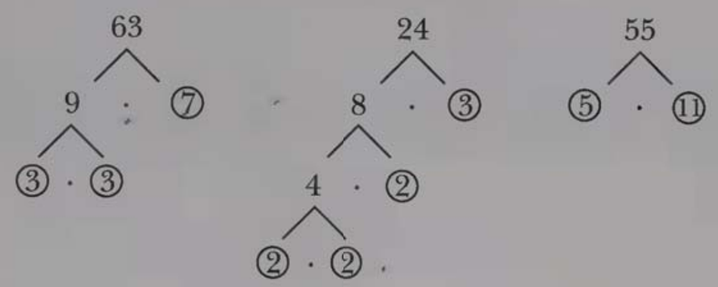
$$\begin{aligned} \text{LCD} &= 2^2 \cdot 3 \cdot 5 \\ &= 4 \cdot 3 \cdot 5 \\ &= 12 \cdot 5 \\ &= 60 \end{aligned}$$

Write all three fractions in terms of the least common denominator, 60.

$$\frac{1 \cdot 15}{4 \cdot 15} = \frac{15}{60} \qquad \frac{2 \cdot 12}{5 \cdot 12} = \frac{24}{60} \qquad \frac{5 \cdot 10}{6 \cdot 10} = \frac{50}{60}$$

8.14 Write the fractions in terms of the least common denominator: $\frac{5}{63}, \frac{17}{24}$, and $\frac{49}{55}$.

Construct factor trees for all three denominators.



Identify the prime factorizations of the denominators, based on the factor trees.

$$63 = 3^2 \cdot 7 \qquad 24 = 2^3 \cdot 3 \qquad 55 = 5 \cdot 11$$

Five unique prime factors are present: 2, 3, 5, 7, and 11. Multiply all five of the unique factors, raising each to its highest power.

$$\begin{aligned} \text{LCD} &= 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \\ &= 8 \cdot 9 \cdot 5 \cdot 7 \cdot 11 \\ &= 72 \cdot 5 \cdot 7 \cdot 11 \\ &= 360 \cdot 7 \cdot 11 \\ &= 2{,}520 \cdot 11 \\ &= 27{,}720 \end{aligned}$$

Write each of the original fractions in terms of the least common denominator, 27,720.

27,720 ÷ 63 = 440
27,720 ÷ 24 = 1,155
27,720 ÷ 55 = 504

$$\frac{5 \cdot 440}{63 \cdot 440} = \frac{2{,}200}{27{,}720} \qquad \frac{17 \cdot 1{,}155}{24 \cdot 1{,}155} = \frac{19{,}635}{27{,}720} \qquad \frac{49 \cdot 504}{55 \cdot 504} = \frac{24{,}696}{27{,}720}$$

Adding and Subtracting Proper Fractions

Add the top, leave the bottom alone

8.15 Simplify the expression: $\frac{1}{7} + \frac{3}{7}$.

You can only add or subtract fractions that share a common denominator. The fractions in this expression share a common denominator of 7, so calculate the sum of the fractions by adding the numerators; that sum should be placed over the common denominator 7.

It doesn't have to be the LEAST common denominator—the denominators just have to match.

$$\frac{1}{7}+\frac{3}{7}=\frac{1+3}{7}$$
$$=\frac{4}{7}$$

Only the numerators are added when you calculate the sum of two fractions—not the denominators.

8.16 Simplify the expression, reducing the answer to lowest terms: $\frac{2}{9}+\frac{4}{9}$.

This is important. Remember this.

The directions in this problem are actually redundant, and future problems will not include the modifying phrase "reducing the answer to lowest terms." When a problem directs you to simplify an expression containing fractions, you should always assume that includes the final step of reducing the answer to lowest terms.

The fractions share a common denominator, so add the numerators and write the answer above the common denominator.

$$\frac{2}{9}+\frac{4}{9}=\frac{2+4}{9}$$
$$=\frac{6}{9}$$

Check out Problems 7.27–7.34 if you need help reducing fractions to lowest terms.

The numerator and denominator of $\frac{6}{9}$ share a common factor: 3. Divide the numerator and denominator by 3 to reduce the fraction to lowest terms.

$$\frac{6\div 3}{9\div 3}=\frac{2}{3}$$

Therefore, $\frac{2}{9}+\frac{4}{9}=\frac{2}{3}$.

8.17 Simplify the expression: $-\frac{7}{6}+\frac{1}{6}$.

When fractions share the same denominator, you can add or subtract the corresponding numerators. Combine the numerators and divide by the common denominator, 6.

$$-\frac{7}{6}+\frac{1}{6}=\frac{-7+1}{6}$$

Notice that the sign that precedes each fraction is transferred to the corresponding numerator. For example, the negative sign in front of $-\frac{7}{6}$ is placed in front of the numerator when the fractions are combined: –7. Similarly, the positive sign that precedes $\frac{1}{6}$ is placed in front of its numerator: +1. Combine the numbers in the numerator and reduce the fraction to lowest terms.

$$=\frac{-6}{6}$$
$$=-1$$

Recall that any number divided by itself is 1; therefore, $6 \div 6 = 1$. Any number divided by its opposite is equal to -1. In this problem, $-6 \div 6 = -1$.

8.18 Simplify the expression: $\frac{2}{3}-\frac{1}{12}$.

You cannot combine the fractions until they share a common denominator. Both denominators divide evenly into the larger denominator, 12. Therefore, 12 is the least common denominator. Construct a fraction equivalent to $\frac{2}{3}$ that has a denominator of 12 by multiplying the numerator and denominator by 4.

$$\frac{2}{3}-\frac{1}{12}=\frac{2\cdot 4}{3\cdot 4}-\frac{1}{12}$$
$$=\frac{8}{12}-\frac{1}{12}$$

This negative sign moves here

Now that both fractions are written in terms of the least common denominator, you can combine the numerators.

$$=\frac{8-1}{12}$$
$$=\frac{7}{12}$$

8.19 Simplify the expression: $\frac{7}{10}-\frac{3}{10}+\frac{2}{5}$.

Two of the fractions share a common denominator, $\frac{7}{10}$ and $-\frac{3}{10}$. Express the remaining fraction in terms of the least common denominator by multiplying its numerator and denominator by 2.

$$\frac{7}{10}-\frac{3}{10}+\frac{2}{5}=\frac{7}{10}-\frac{3}{10}+\frac{2\cdot 2}{5\cdot 2}$$
$$=\frac{7}{10}-\frac{3}{10}+\frac{4}{10}$$

Combine the numerators of the fractions.

$$=\frac{7-3+4}{10}$$
$$=\frac{4+4}{10}$$
$$=\frac{8}{10}$$

Reduce the fraction to lowest terms by dividing the numerator and denominator by the greatest common factor, 2.

$$= \frac{8 \div 2}{10 \div 2}$$
$$= \frac{4}{5}$$

Multiply the top and bottom of $-\frac{4}{5}$ by 4, and multiply the top and bottom of $\frac{1}{4}$ by 5. This is very similar to Problem 8.7, so look there if you need more help.

8.20 Simplify the expression: $\frac{3}{20} - \frac{4}{5} + \frac{1}{4}$.

The largest denominator in this expression is also the least common denominator, because both of the smaller denominators divide into it evenly: $20 \div 5 = 4$ and $20 \div 4 = 5$. Rewrite the expression in terms of the least common denominator.

$$\frac{3}{20} - \frac{4}{5} + \frac{1}{4} = \frac{3}{20} - \frac{4 \cdot 4}{5 \cdot 4} + \frac{1 \cdot 5}{4 \cdot 5}$$
$$= \frac{3}{20} - \frac{16}{20} + \frac{5}{20}$$

Now that all three fractions share a common denominator, combine the values in the numerator.

$$= \frac{3 - 16 + 5}{20}$$
$$= \frac{-13 + 5}{20}$$
$$= \frac{-8}{20}$$

Reduce the fraction to lowest terms to complete the problem.

$$= \frac{-8 \div 4}{20 \div 4}$$
$$= -\frac{2}{5}$$

8.21 Simplify the expression: $\frac{1}{6} + \frac{4}{9}$.

The larger denominator in this expression (9) is not the least common denominator, because it is not evenly divisible by the smaller denominator (6). Apply the method described in the solution to Problem 8.1: Investigate mutliples of the larger denominator until you identify a number that is divisible by the smaller denominator. In this case, simply multiplying the larger denominator by 2 generates the least common denominator.

$$9 \cdot 2 = 18$$

Write both fractions in terms of the least common denominator.

$$\begin{aligned}\frac{1}{6}+\frac{4}{9}&=\frac{1\cdot 3}{6\cdot 3}+\frac{4\cdot 2}{9\cdot 2}\\&=\frac{3}{18}+\frac{8}{18}\end{aligned}$$

Add the numerators and divide by the common denominator.

$$=\frac{11}{18}$$

Because 11 is a prime number, and 11 is not a factor of 18, the fraction cannot be reduced any further.

8.22 Simplify the expression: $\frac{9}{10}-\frac{11}{12}$.

List multiples of the larger denominator until you identify one that is evenly divisible by the smaller denominator.

$$\begin{aligned}12\cdot 2&=24\\12\cdot 3&=36\\12\cdot 4&=48\\12\cdot 5&=60\end{aligned}$$

The least common denominator is 60, as it is the smallest positive integer that is evenly divisible by 10 and 12. Rewrite the expression in terms of the least common denominator.

$$\begin{aligned}\frac{9}{10}-\frac{11}{12}&=\frac{9\cdot 6}{10\cdot 6}-\frac{11\cdot 5}{12\cdot 5}\\&=\frac{54}{60}-\frac{55}{60}\end{aligned}$$

The second fraction is subtracted from the first, so the second numerator should be subtracted from the first numerator.

$$\begin{aligned}&=\frac{54-55}{60}\\&=\frac{-1}{60}\end{aligned}$$

If there are negative signs in the numerator AND denominator, then you can erase both of them, because the fraction is positive! A negative divided by a negative is a positive.

You can place the negative sign in the numerator, as written above, or in the denominator: $\frac{1}{-60}$; the placement of the negative sign does not affect the value of the fraction. However, it is most common to place the negative sign in front of the fraction, as demonstrated below.

$$=-\frac{1}{60}$$

8.23 Simplify the expression: $\frac{1}{8}-\frac{5}{6}$.

List multiples of the larger denominator until you identify a product that is divisible by the smaller denominator.

$$8\cdot 2=16$$
$$8\cdot 3=24$$

Rewrite the expression in terms of the least common denominator, 24.

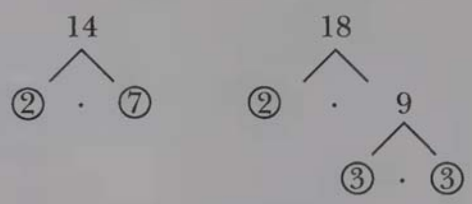
$$\begin{aligned}\frac{1}{8}-\frac{5}{6}&=\frac{1\cdot 3}{8\cdot 3}-\frac{5\cdot 4}{6\cdot 4}\\&=\frac{3}{24}-\frac{20}{24}\end{aligned}$$

Combine the numerators of the fractions and divide by the common denominator.

$$\begin{aligned}&=\frac{3-20}{24}\\&=\frac{-17}{24}\\&=-\frac{17}{24}\end{aligned}$$

8.24 Simplify the expression: $\frac{3}{14}+\frac{5}{18}$.

Apply the technique described in Problem 8.2 to calculate the least common denominator. Begin by constructing factor trees for the denominators of the expression in order to identify their prime factorizations.



The prime factorizations of the denominators are $14 = 2 \cdot 7$ and $18 = 2 \cdot 3^2$. The least common denominator is the product of all three unique factors (2, 3, and 7) when each is raised to its highest power.

$$\begin{aligned}\text{LCD}&=2\cdot 3^2\cdot 7\\&=2\cdot 9\cdot 7\\&=18\cdot 7\\&=126\end{aligned}$$

Note that $126 \div 14 = 9$ and $126 \div 18 = 7$. Use these quotients to rewrite the expression in terms of the least common denominator.

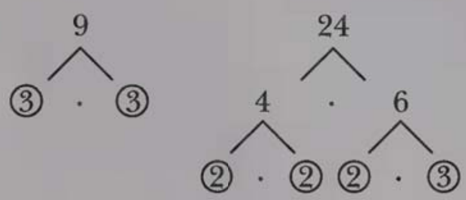
$$\begin{aligned}\frac{3}{14}+\frac{5}{18}&=\frac{3\cdot 9}{14\cdot 9}+\frac{5\cdot 7}{18\cdot 7}\\&=\frac{27}{126}+\frac{35}{126}\end{aligned}$$

Combine the fractions and reduce the result to lowest terms.

$$\begin{aligned}&=\frac{27+35}{126}\\&=\frac{62}{126}\\&=\frac{62\div 2}{126\div 2}\\&=\frac{31}{63}\end{aligned}$$

8.25 Simplify the expression: $\frac{7}{24}+\frac{2}{9}$.

Construct factor trees to identify the prime factorizations of the denominators.



Once your denominators get close to (or are greater than) 20, you should probably use the factor tree method to calculate the LCD. The multiples method will probably take too long.

The prime factorizations are $9 = 3^2$ and $24 = 2^3 \cdot 3$. Use the highest powers of the prime factors to calculate the least common denominator.

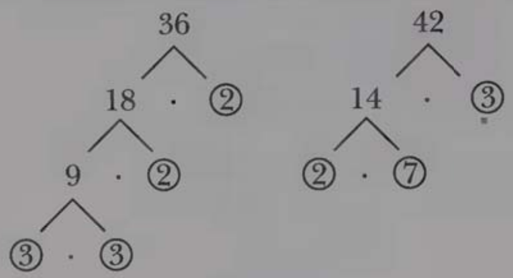
$$\begin{aligned}\text{LCD}&=2^3\cdot 3^2\\&=8\cdot 9\\&=72\end{aligned}$$

Write the original expression in terms of the least common denominator.

$$\begin{aligned}\frac{7}{24}+\frac{2}{9}&=\frac{7\cdot 3}{24\cdot 3}+\frac{2\cdot 8}{9\cdot 8}\\&=\frac{21}{72}+\frac{16}{72}\\&=\frac{37}{72}\end{aligned}$$

8.26 Simplify the expression: $\frac{29}{36} - \frac{37}{42}$.

Construct factor trees to identify the prime factorizations of 36 and 42.



According to the figure, $36 = 2^2 \cdot 3^2$ and $42 = 2 \cdot 3 \cdot 7$. Calculate the least common denominator.

$$\begin{aligned} \text{LCD} &= 2^2 \cdot 3^2 \cdot 7 \\ &= 4 \cdot 9 \cdot 7 \\ &= 36 \cdot 7 \\ &= 252 \end{aligned}$$

Notice that $252 \div 36 = 7$ and $252 \div 42 = 6$. Use these quotients to write the original expression in terms of the least common denominator.

$$\begin{aligned} \frac{29}{36} - \frac{37}{42} &= \frac{29 \cdot 7}{36 \cdot 7} - \frac{37 \cdot 6}{42 \cdot 6} \\ &= \frac{203}{252} - \frac{222}{252} \end{aligned}$$

Combine the fractions.

$$\begin{aligned} &= \frac{203 - 222}{252} \\ &= \frac{-19}{252} \\ &= -\frac{19}{252} \end{aligned}$$

The numerator is prime and is not a factor of the denominator, so the fraction cannot be reduced any further.

8.27 Simplify the expression: $\frac{5}{9} - \frac{2}{3} + \frac{1}{4}$.

List multiples of the largest denominator (9) until the product is divisible by both of the smaller denominators (3 and 4).

$$\begin{aligned} 9 \cdot 2 &= 18 \\ 9 \cdot 3 &= 27 \\ 9 \cdot 4 &= 36 \end{aligned}$$

Note that $36 \div 9 = 4$, $36 \div 3 = 12$, and $36 \div 4 = 9$. Use these quotients to rewrite the expression in terms of the least common denominator, 36.

$$\begin{aligned}\frac{5}{9}-\frac{2}{3}+\frac{1}{4}&=\frac{5\cdot 4}{9\cdot 4}-\frac{2\cdot 12}{3\cdot 12}+\frac{1\cdot 9}{4\cdot 9}\\&=\frac{20}{36}-\frac{24}{36}+\frac{9}{36}\end{aligned}$$

Combine the numerators and ensure that the resulting fraction is reduced to lowest terms.

$$\begin{aligned}&=\frac{20-24+9}{36}\\&=\frac{-4+9}{36}\\&=\frac{5}{36}\end{aligned}$$

8.28 Simplify the expression: $\frac{3}{16}+\frac{1}{2}-\frac{5}{12}$.

List multiples of the largest denominator (16) until the product is divisible by both of the smaller denominators (2 and 12).

$$\begin{aligned}16\cdot 2&=32\\16\cdot 3&=48\end{aligned}$$

Note that $48 \div 16 = 3$, $48 \div 2 = 24$, and $48 \div 12 = 4$. Use these quotients to write the expression in terms of the least common denominator, 48.

$$\begin{aligned}\frac{3}{16}+\frac{1}{2}-\frac{5}{12}&=\frac{3\cdot 3}{16\cdot 3}+\frac{1\cdot 24}{2\cdot 24}-\frac{5\cdot 4}{12\cdot 4}\\&=\frac{9}{48}+\frac{24}{48}-\frac{20}{48}\end{aligned}$$

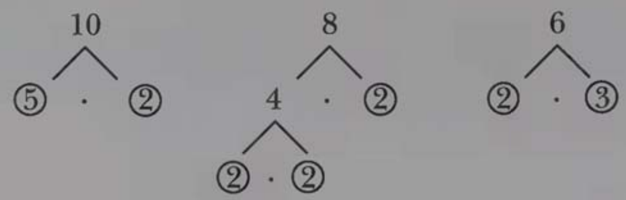
Combine the numerators and ensure the resulting fraction is reduced to lowest terms.

$$\begin{aligned}&=\frac{9+24-20}{48}\\&=\frac{33-20}{48}\\&=\frac{13}{48}\end{aligned}$$

8.29 Simplify the expression: $\frac{7}{10}+\frac{3}{8}-\frac{1}{6}$.

Construct factor trees for the denominators in order to identify their prime factorizations.

If you list multiples of 10, it's going to take you a long time to get to the LCD.



The prime factorizations of the denominators are $6 = 2 \cdot 3$, $8 = 2^3$, and $10 = 2 \cdot 5$. Raise each of the prime factors (2, 3, and 5) to its highest power and calculate the product to determine the least common denominator.

$$\begin{aligned} \text{LCD} &= 2^3 \cdot 3 \cdot 5 \\ &= 8 \cdot 3 \cdot 5 \\ &= 24 \cdot 5 \\ &= 120 \end{aligned}$$

Note that $120 \div 10 = 12$, $120 \div 8 = 15$, and $120 \div 6 = 20$. Use these quotients to express the fractions in terms of the least common denominator, 120, and combine the fractions once they all share the same denominator.

$$\begin{aligned} \frac{7}{10} + \frac{3}{8} - \frac{1}{6} &= \frac{7 \cdot 12}{10 \cdot 12} + \frac{3 \cdot 15}{8 \cdot 15} - \frac{1 \cdot 20}{6 \cdot 20} \\ &= \frac{84}{120} + \frac{45}{120} - \frac{20}{120} \\ &= \frac{84 + 45 - 20}{120} \\ &= \frac{129 - 20}{120} \\ &= \frac{109}{120} \end{aligned}$$

8.30 Simplify the expression: $\frac{4}{9} - \frac{5}{14} - \frac{19}{35}$.

Identify the prime factorizations of the denominators.

$$\begin{aligned} 9 &= 3^2 \\ 14 &= 2 \cdot 7 \\ 35 &= 5 \cdot 7 \end{aligned}$$

Calculate the least common denominator by multiplying the highest powers of the prime factors identified above: 2, 3, 5, and 7.

$$\begin{aligned} \text{LCD} &= 2 \cdot 3^2 \cdot 5 \cdot 7 \\ &= 2 \cdot 9 \cdot 5 \cdot 7 \\ &= 18 \cdot 5 \cdot 7 \\ &= 90 \cdot 7 \\ &= 630 \end{aligned}$$

Divide the least common denominator by each of the individual denominators: $630 \div 9 = 70$, $630 \div 14 = 45$, and $630 \div 35 = 18$. Use the quotients to write the fractions in terms of the least common denominator.

$$\begin{aligned}\frac{4}{9}-\frac{5}{14}-\frac{19}{35} &= \frac{4\cdot 70}{9\cdot 70}-\frac{5\cdot 45}{14\cdot 45}-\frac{19\cdot 18}{35\cdot 18}\\ &= \frac{280}{630}-\frac{225}{630}-\frac{342}{630}\\ &= \frac{280-225-342}{630}\\ &= \frac{55-342}{630}\\ &= -\frac{287}{630}.\end{aligned}$$

Divisibility Test for 7: Take off the last digit, double it, and subtract it from the rest of the number to see if you get something divisible by 7.

287: 28 − 2(7) = 28 − 14 = 14
630: 63 − 2(0) = 63 − 0 = 63

The numerator and denominator are divisible by 7; reduce the fraction to lowest terms.

$$\begin{aligned} &= -\frac{287\div 7}{630\div 7}\\ &= -\frac{41}{90}\end{aligned}$$

8.31 Simplify the expression: $\frac{7}{27}-\frac{50}{99}+\frac{1}{12}-\frac{19}{30}$.

Identify the prime factorizations of all four denominators, using a factor tree if necessary.

$$\begin{aligned}27 &= 3^3\\ 99 &= 3^2\cdot 11\\ 12 &= 2^2\cdot 3\\ 30 &= 2\cdot 3\cdot 5\end{aligned}$$

Compute the least common denominator.

$$\begin{aligned}\text{LCD} &= 2^2\cdot 3^3\cdot 5\cdot 11\\ &= 4\cdot 27\cdot 5\cdot 11\\ &= 108\cdot 5\cdot 11\\ &= 540\cdot 11\\ &= 5{,}940\end{aligned}$$

Divide the least common denominator by each of the denominators in the expression.

$$5{,}940 \div 27 = 220$$
$$5{,}940 \div 99 = 60$$
$$5{,}940 \div 12 = 495$$
$$5{,}940 \div 30 = 198$$

Use the quotients listed above to write the expression in terms of the least common denominator, 5,940.

$$\begin{aligned}\frac{7}{27}-\frac{50}{99}+\frac{1}{12}-\frac{19}{30} &= \frac{7\cdot 220}{27\cdot 220}-\frac{50\cdot 60}{99\cdot 60}+\frac{1\cdot 495}{12\cdot 495}-\frac{19\cdot 198}{30\cdot 198}\\ &= \frac{1{,}540}{5{,}940}-\frac{3{,}000}{5{,}940}+\frac{495}{5{,}940}-\frac{3{,}762}{5{,}940}\\ &= \frac{1{,}540-3{,}000+495-3{,}762}{5{,}940}\\ &= \frac{-1{,}460+495-3{,}762}{5{,}940}\\ &= \frac{-965-3{,}762}{5{,}940}\\ &= -\frac{4{,}727}{5{,}940}\end{aligned}$$

Although the numerator and denominator are relatively large numbers, they do not share any common factors, so the fraction is, by default, already reduced to lowest terms.

Improper Fractions and Mixed Numbers

Add and subtract weird looking fractions

Note: In Problems 8.32–8.33, you use different techniques to simplify the expression $\frac{8}{3}+\frac{14}{9}$.

8.32 Combine the improper fractions after writing them in terms of the least common denominator.

There is no difference in the techniques used to add proper and improper fractions; like Problems 8.15–8.31, improper fractions require only a common denominator before they are added or subtracted. In this case, 9 is the least common denominator, so you should identify a fraction equivalent to $\frac{8}{3}$ that has a denominator of 9.

$$\begin{aligned}\frac{8}{3}+\frac{14}{9} &= \frac{8\cdot 3}{3\cdot 3}+\frac{14}{9}\\ &= \frac{24}{9}+\frac{14}{9}\end{aligned}$$

The fractions now share a common denominator, so their numerators can be combined.

$$=\frac{24+14}{9}$$
$$=\frac{38}{9}$$

Most teachers want you to leave improper fraction answers alone, instead of converting them to mixed numbers. Problem 8.33 shows you why mixed numbers are harder to work with.

Note: In Problems 8.32–8.33, you use different techniques to simplify the expression $\frac{8}{3}+\frac{14}{9}$.

8.33 Convert the improper fractions to mixed numbers and add them, expressing the answer as a mixed number.

Apply the technique described in Problem 7.11 to express the improper fractions as mixed numbers: If each numerator is divided by the corresponding denominator, the quotient is the whole part of the mixed number, the remainder is the numerator of the proper fraction, and the denominator of the proper fraction matches the denominator of the original improper fraction.

$$\frac{8}{3}=2\frac{2}{3}, \text{ because } 8\div 3=2 \text{ r } 2$$
$$\frac{14}{9}=1\frac{5}{9}, \text{ because } 14\div 9=1 \text{ r } 5$$

Rewrite the original expression, substituting equivalent mixed numbers for the improper fractions.

$$\frac{8}{3}+\frac{14}{9}=2\frac{2}{3}+1\frac{5}{9}$$

Recall that mixed numbers are the sum of whole numbers and proper fractions, though the addition sign that separates the whole numbers and proper fractions is omitted. Insert the implied addition symbols.

$$=2+\frac{2}{3}+1+\frac{5}{9}$$

Regroup the expression, separating the whole numbers from the proper fractions.

$$=(2+1)+\left(\frac{2}{3}+\frac{5}{9}\right)$$

Rewrite the proper fractions in terms of the least common denominator, 9.

$$=(2+1)+\left(\frac{2\cdot 3}{3\cdot 3}+\frac{5}{9}\right)$$
$$=(2+1)+\left(\frac{6}{9}+\frac{5}{9}\right)$$

Add the whole and fractional parts independently.

$$=(2+1)+\left(\frac{6+5}{9}\right)$$
$$=3+\frac{11}{9}$$

The problem wants the answer in the form of a mixed number, which can't contain an improper fraction.

Rewrite $\frac{11}{9}$ as a mixed number: $\frac{11}{9}=1\frac{2}{9}$. Substitute the mixed number into the expression.

$$=3+1\frac{2}{9}$$
$$=3+1+\frac{2}{9}$$
$$=4\frac{2}{9}$$

If you convert the answer above into an improper fraction, the result exactly matches the solution to Problem 8.32. Thus, the technique that you use to add or subtract improper fractions and mixed numbers does not affect the final sum or difference.

$$4\frac{2}{9}=\frac{9\cdot 4+2}{9}$$
$$=\frac{36+2}{9}$$
$$=\frac{38}{9}$$

Note: In Problems 8.34–8.35, you use different techniques to simplify $\frac{27}{8}-\frac{33}{14}$.

8.34 Combine the improper fractions after writing them in terms of the least common denominator.

List multiples of the larger denominator until the product is divisible by the smaller denominator.

$$14\cdot 2=28$$
$$14\cdot 3=42$$
$$14\cdot 4=56$$

Write both fractions in terms of the least common denominator, 56.

$$\begin{aligned}\frac{27}{8}-\frac{33}{14}&=\frac{27\cdot 7}{8\cdot 7}-\frac{33\cdot 4}{14\cdot 4}\\&=\frac{189}{56}-\frac{132}{56}\\&=\frac{189-132}{56}\\&=\frac{57}{56}\end{aligned}$$

Note: In Problems 8.34–8.35, you use different techniques to simplify $\frac{27}{8}-\frac{33}{14}$.

8.35 Convert the improper fractions to mixed numbers and combine them, expressing the answer as a mixed number.

Convert the improper fractions to mixed numbers and insert the implied signs between the whole number and proper fraction portions of the mixed numbers.

$$\begin{aligned}\frac{27}{8}-\frac{33}{14}&=3\frac{3}{8}-2\frac{5}{14}\\&=3+\frac{3}{8}-2-\frac{5}{14}\end{aligned}$$

Notice that a negative sign appears before 2 and before $\frac{5}{14}$. In the mixed number expression, the entire fraction $2\frac{5}{14}$ is subtracted. Therefore, the whole number 2 and the proper fraction $\frac{5}{14}$ should be subtracted when the parts of the whole number are separated. Combine the whole numbers and write the fractions in terms of the least common denominator, 56.

$$\begin{aligned}&=(3-2)+\left(\frac{3}{8}-\frac{5}{14}\right)\\&=(3-2)+\left(\frac{3\cdot 7}{8\cdot 7}-\frac{5\cdot 4}{14\cdot 4}\right)\\&=(3-2)+\left(\frac{21}{56}-\frac{20}{56}\right)\\&=(3-2)+\left(\frac{21-20}{56}\right)\\&=1+\frac{1}{56}\\&=1\frac{1}{56}\end{aligned}$$

If you convert this mixed number to an improper fraction, it exactly matches the solution to Problem 8.34.

$$\begin{aligned} 1\frac{1}{56} &= \frac{56\cdot 1+1}{56} \\ &= \frac{56+1}{56} \\ &= \frac{57}{56} \end{aligned}$$

Note: In Problems 8.36–8.37, you use different techniques to simplify $4+\frac{17}{5}$.

8.36 Express 4 as an improper fraction and combine the fractions using the least common denominator.

Recall that any number divided by one is equal to that number. Therefore, any fraction with a denominator of one is equal to the integer that appears in the numerator. In this problem, $4=\frac{4}{1}$. Rewrite the expression, replacing 4 with the equivalent fraction.

$$4+\frac{17}{5}=\frac{4}{1}+\frac{17}{5}$$

All real numbers are divisible by 1.

The larger of the two denominators (5) is also the least common denominator, because it is divisible by the smaller denominator (1). Write the expression in terms of the least common denominator and simplify the expression.

$$\begin{aligned} &= \frac{4\cdot 5}{1\cdot 5}+\frac{17}{5} \\ &= \frac{20}{5}+\frac{17}{5} \\ &= \frac{20+17}{5} \\ &= \frac{37}{5} \end{aligned}$$

Note: In Problems 8.36–8.37, you use different techniques to simplify $4+\frac{17}{5}$.

8.37 Convert the improper fraction to a mixed number and simplify the expression, writing the answer as a mixed number.

Express $\frac{17}{5}$ as a mixed number.

$$\frac{17}{5}=3\frac{2}{5}$$

Replace the improper fraction in the expression with the equivalent mixed number.

$$4+\frac{17}{5}=4+3\frac{2}{5}$$

To calculate the sum, add the whole numbers in the expression.

$$\begin{aligned}&=(4+3)+\frac{2}{5}\\&=7\frac{2}{5}\end{aligned}$$

If you convert the answer to an improper fraction, it exactly matches the solution to Problem 8.36.

$$\begin{aligned}7\frac{2}{5}&=\frac{5\cdot 7+2}{5}\\&=\frac{35+2}{5}\\&=\frac{37}{5}\end{aligned}$$

Whereas it is usually easier to add improper fractions than it is to add mixed numbers (as demonstrated in Problems 8.32–8.35), it is slightly easier to add integers to mixed numbers than to add integers to improper fractions.

8.38 Simplify the expression: $6\frac{1}{8}+\frac{51}{12}$.

Rewrite the expression, converting the mixed number to an improper fraction.

$$\begin{aligned}6\frac{1}{8}+\frac{51}{12}&=\frac{8\cdot 6+1}{8}+\frac{51}{12}\\&=\frac{48+1}{8}+\frac{51}{12}\\&=\frac{49}{8}+\frac{51}{12}\end{aligned}$$

Multiply the larger denominator by 2 and you get $12\cdot 2=24$, which is also a multiple of 8, so 24 is the LCD.

Rewrite the expression in terms of the least common denominator (24) and combine the fractions.

$$\begin{aligned}&=\frac{49\cdot 3}{8\cdot 3}+\frac{51\cdot 2}{12\cdot 2}\\&=\frac{147}{24}+\frac{102}{24}\\&=\frac{147+102}{24}\\&=\frac{249}{24}\end{aligned}$$

Reduce the fraction to lowest terms.

$$=\frac{249\div 3}{24\div 3}$$
$$=\frac{83}{8}$$

8.39 Simplify the expression: $\frac{37}{24}-3\frac{5}{18}$.

Convert the mixed number to an improper fraction.

$$\begin{aligned}\frac{37}{24}-3\frac{5}{18}&=\frac{37}{24}-\frac{18\cdot 3+5}{18}\\&=\frac{37}{24}-\frac{54+5}{18}\\&=\frac{37}{24}-\frac{59}{18}\end{aligned}$$

Write the fractions in terms of the least common denominator, 72, and combine them.

$$\begin{aligned}&=\frac{37}{24}\cdot\frac{3}{3}-\frac{59}{18}\cdot\frac{4}{4}\\&=\frac{111}{72}-\frac{236}{72}\\&=-\frac{125}{72}\end{aligned}$$

8.40 Simplify the expression: $\frac{92}{27}-\frac{47}{18}+6$.

Identify the prime factorizations of the denominators. Note that you can express the integer 6 as the fraction $\frac{6}{1}$, but there is no need to include the denominator 1 when you calculate the least common denominator.

$$27=3^3$$
$$18=2\cdot 3^2$$

Compute the least common denominator.

$$\begin{aligned}\text{LCD}&=2\cdot 3^3\\&=2\cdot 27\\&=54\end{aligned}$$

Rewrite the expression in terms of the least common denominator.

$$\begin{aligned}\frac{92}{27}-\frac{47}{18}+6&=\frac{92\cdot 2}{27\cdot 2}-\frac{47\cdot 3}{18\cdot 3}+\frac{6\cdot 54}{1\cdot 54}\\&=\frac{184}{54}-\frac{141}{54}+\frac{324}{54}\end{aligned}$$

The denominator is 1, so multiply the top and bottom of the fraction by the LCD (54).

Combine the fractions.

$$\begin{aligned}&=\frac{184-141+324}{54}\\&=\frac{43+324}{54}\\&=\frac{367}{54}\end{aligned}$$

The fraction cannot be reduced any further, as 367 and 54 share no common factors (apart from the trivial factor 1 that is shared by all integers). Note that answers are typically expressed as improper fractions rather than mixed numbers. Under no circumstances should you convert a fractional answer to a decimal unless specifically instructed. (This process is described in Chapter 11.)

8.41 Simplify the expression: $\frac{17}{5}+\frac{19}{12}-5\frac{3}{28}$.

Express the mixed number as an improper fraction.

$$\begin{aligned}5\frac{3}{28}&=\frac{28\cdot 5+3}{28}\\&=\frac{140+3}{28}\\&=\frac{143}{28}\end{aligned}$$

Identify the prime factorizations of the denominators 5, 12, and 28.

$$\begin{aligned}5&=5\\12&=2^2\cdot 3\\28&=2^2\cdot 7\end{aligned}$$

Compute the least common denominator.

$$\begin{aligned}\text{LCD}&=2^2\cdot 3\cdot 5\cdot 7\\&=4\cdot 3\cdot 5\cdot 7\\&=12\cdot 5\cdot 7\\&=60\cdot 7\\&=420\end{aligned}$$

Write the expression in terms of the least common denominator. Notice that the mixed number in the original expression is replaced by the improper fraction calculated in a previous step.

$$\begin{aligned}\frac{17}{5}+\frac{19}{12}-5\frac{3}{28}&=\frac{17}{5}+\frac{19}{12}-\frac{143}{28}\\&=\frac{17\cdot 84}{5\cdot 84}+\frac{19\cdot 35}{12\cdot 35}-\frac{143\cdot 15}{28\cdot 15}\\&=\frac{1,428}{420}+\frac{665}{420}-\frac{2,145}{420}\end{aligned}$$

Combine the fractions.

$$\begin{aligned}&=\frac{1,428+665-2,145}{420}\\&=\frac{2,093-2,145}{420}\\&=-\frac{52}{420}\end{aligned}$$

Reduce the fraction to lowest terms.

$$\begin{aligned}&=-\frac{52\div 4}{420\div 4}\\&=-\frac{13}{105}\end{aligned}$$

Chapter 9

MULTIPLYING AND DIVIDING FRACTIONS

Much easier than adding and subtracting them

The process of adding and subtracting fractions, investigated in substantial detail in Chapter 8, is straightforward but rarely simple. To complicate the matter, the larger the denominators of the fractions involved, the more complex it is to combine those fractions. The processes of multiplying and dividing fractions, on the other hand, are simple in concept and practice. Furthermore, the sizes of the denominators (or for that matter, the numerators) do not affect the difficulty of the problems in a substantial way.

This chapter begins by exploring proper fraction multiplication and division, graduates to the manipulation of improper fractions and mixed numbers, and concludes with a brief study of complex fractions—fractions whose numerators and/or denominators consist of other fractions.

Chapter 8 was all about the least common denominator. Honestly. LCD this, LCD that. LCD, LCD, LCD. You'll barely read about least common denominators in this chapter. In fact, the only time you'll see them at all is when the book sneaks an addition or subtraction problem into the mix. If I say too much more, I'll spoil the only interesting concepts in the book, so allow me to be coy and just tease some of the things you'll learn in this chapter:

- You can leave denominators alone when you multiply and divide fractions!
- Dividing fractions works just like multiplying fractions, just with a twist.
- You can reduce fraction products before you get to the final answer!
- Fractions inside fractions? No problem.

Some good news: As tricky and time-consuming as Chapter 8 was, Chapter 9 is much easier and more interesting. (Did I mention that you'll be calculating a lot fewer least common denominators?)

Multiplying Proper Fractions

Top times top, bottom times bottom

9.1 Explain how denominators are treated differently when multiplying fractions versus adding fractions.

Whereas you can only add fractions that are expressed in terms of a common denominator, no such requirement exists for the multiplication of fractions. A very simple procedure, described in Problem 9.2, allows you to calculate the product of any two fractions, regardless of the numbers in the numerators and denominators.

Another key difference is the treatment of the denominator. When fractions are added, only their numerators are combined, and the result is recorded over the common denominator. However, when fractions are multiplied, the numerators *and* the denominators are both involved in the computation of the product.

Multiply the top numbers and the bottom numbers separately. The answer is top times top over bottom times bottom.

9.2 Simplify the expression: $\frac{4}{7}\cdot\frac{2}{5}$.

The product of two or more fractions is the product of the numerators divided by the product of the denominators.

$$\begin{aligned}\frac{4}{7}\cdot\frac{2}{5}&=\frac{4\cdot 2}{7\cdot 5}\\&=\frac{8}{35}\end{aligned}$$

9.3 Simplify the expression: $\frac{7}{9}\cdot\frac{3}{4}$.

Apply the technique described in Problem 9.2: The product of two fractions is equal to the product of their numerators divided by the product of their denominators.

$$\begin{aligned}\frac{7}{9}\cdot\frac{3}{4}&=\frac{7\cdot 3}{9\cdot 4}\\&=\frac{21}{36}\end{aligned}$$

Reduce the fraction to lowest terms.

$$\begin{aligned}&=\frac{21\div 3}{36\div 3}\\&=\frac{7}{12}\end{aligned}$$

Note: Problems 9.4–9.5 describe how to reduce a product of fractions while multiplying those fractions.

9.4 Explain why the product $\frac{5}{8} \cdot \frac{4}{7}$ will need to be reduced to lowest terms.

When two or more fractions are multiplied, you can determine whether the product will be expressed in lowest terms by analyzing the individual numbers in the numerator and denominator. If any term in the numerator shares a common factor with any term in the denominator, then that common factor will need to be divided out of the numerator and denominator of the product in order to reduce that product to lowest terms.

Write the product $\frac{5}{8} \cdot \frac{4}{7}$ as a single fraction, but do not yet multiply the numbers in the numerator and denominator.

$$\frac{5}{8} \cdot \frac{4}{7} = \frac{5 \cdot 4}{8 \cdot 7}$$

The numerator contains two terms: 5 and 4; the denominator contains two terms as well: 8 and 7. Notice that 4 (in the numerator) and 8 (in the denominator) are both evenly divisible by 4. Thus, to reduce the product to lowest terms, you will have to divide the numerator and denominator by 4. Verify this by simplifying the expression.

$$\begin{aligned}\frac{5 \cdot 4}{8 \cdot 7} &= \frac{20}{56} \\ &= \frac{20 \div 4}{56 \div 4} \\ &= \frac{5}{14}\end{aligned}$$

Note: Problems 9.4–9.5 describe how to reduce a product of fractions while multiplying those fractions.

9.5 Reduce the product $\frac{5}{8} \cdot \frac{4}{7}$ to lowest terms before multiplying the fractions.

As Problem 9.4 explains, if a value in any numerator shares a common factor with a value in any denominator, that common factor will need to be addressed once the fractions are multiplied. Rather than wait until the end of the multiplication process, like Problem 9.4, you may choose to eliminate the common factor as the first step in the process.

Consider the product $\frac{5}{8}\cdot\frac{4}{7}$. Notice that the values 4 in the numerator and 8 in the denominator share the common factor 4, because 4 divides into 4 and 8 without any remainder. Divide both numbers by the common factor: $4 \div 4 = 1$ and $8 \div 4 = 2$. Then, draw a slash through 4 and 8 in the expression to indicate that you are reducing those numbers by a common factor and replace them with the quotients you calculated (2 and 1, respectively), as demonstrated below.

The informal name for this process is "cancelling out" the common factor, because you're physically crossing out a number and replacing it with another number. Most math teachers don't like that terminology, though. They prefer "reducing."

$$\begin{aligned}\frac{5}{8}\cdot\frac{4}{7} &= \frac{5}{\cancel{8}\,2}\cdot\frac{\cancel{4}\,1}{7}\\ &= \frac{5}{2}\cdot\frac{1}{7}\end{aligned}$$

When you calculate this product, the result exactly matches the solution to Problem 9.4.

$$\begin{aligned}&= \frac{5\cdot 1}{2\cdot 7}\\ &= \frac{5}{14}\end{aligned}$$

9.6 Reduce the product $\frac{14}{39}\cdot\frac{13}{28}$ to lowest terms before multiplying the fractions.

If you do not reduce before multiplying, the numerator and denominator of the product will be large, and a fair amount of work will be required to reduce the answer to lowest terms. Identify pairs of numbers that share a common factor. Be mindful of one restriction: One of the numbers must appear in a numerator and one must appear in a denominator.

Do you feel like reducing $\frac{182}{1,092}$? I don't.

For instance, 14 from the numerator and 28 from the denominator are both divisible by 14. Divide each of them by the common factor: $14 \div 14 = 1$ and $28 \div 14 = 2$. Rewrite the expression, replacing 14 and 28 with these quotients.

$$\begin{aligned}\frac{14}{39}\cdot\frac{13}{28} &= \frac{\cancel{14}\,1}{39}\cdot\frac{13}{\cancel{28}\,2}\\ &= \frac{1}{39}\cdot\frac{13}{2}\end{aligned}$$

The product is not yet fully reduced. Notice that the numerator value 13 and the denominator value 39 are both divisible by 13, as $39 \div 13 = 3$ and $13 \div 13 = 1$. Replace 13 and 39 with those quotients.

$$\begin{aligned}&= \frac{1}{\cancel{39}\,3}\cdot\frac{\cancel{13}\,1}{2}\\ &= \frac{1}{3}\cdot\frac{1}{2}\end{aligned}$$

This product is much easier to calculate, and you do not have to reduce the result to lowest terms.

$$=\frac{1}{6}$$

9.7 Reduce the product $\frac{77}{45}\cdot\frac{6}{11}$ to lowest terms before multiplying the fractions.

The numerator value 77 and the denominator value 11 are evenly divisible by 11, so divide both of the numbers by 11 and replace them by the corresponding quotient.

$$\frac{77}{45}\cdot\frac{6}{11}=\frac{\cancel{77}\,7}{45}\cdot\frac{6}{\cancel{11}\,1}$$
$$=\frac{7}{45}\cdot\frac{6}{1}$$

The numerator value 6 and denominator value 45 share a common factor as well: 3. Divide those values by 3 and replace them with the correct quotients.

You can reduce both pairs at the same time if you are feeling confident, instead of doing it in two separate steps. It might look messy with all of the original numbers crossed out at the same time, but if that doesn't confuse you, go for it.

$$=\frac{7}{\cancel{45}\,15}\cdot\frac{\cancel{6}\,2}{1}$$
$$=\frac{7}{15}\cdot\frac{2}{1}$$

Compute the product of the fractions.

$$=\frac{7\cdot 2}{15\cdot 1}$$
$$=\frac{14}{15}$$

9.8 Simplify the expression: $\frac{7}{12}\cdot 4$.

Any integer is equivalent to the fraction with that integer in the numerator and a denominator of one. In this problem, $4=\frac{4}{1}$. Rewrite the expression as the product of two fractions.

$$\frac{7}{12}\cdot 4=\frac{7}{12}\cdot\frac{4}{1}$$

While you can now multiply the numerators and divide by the product of the denominators, it is worthwhile to note that a number in the numerator (4) and a number in the denominator (12) share a common factor (4).

Divide 4 and 12 by 4 and replace those numbers with the quotients: $4 \div 4 = 1$ and $12 \div 4 = 3$.

$$=\frac{7}{\cancel{12}\,3}\cdot\frac{\cancel{4}\,1}{1}$$
$$=\frac{7}{3}\cdot\frac{1}{1}$$
$$=\frac{7\cdot 1}{3\cdot 1}$$
$$=\frac{7}{3}$$

9.9 Translate the following phrase into a product of fractions, predict the answer, and calculate the product to determine the accuracy of your prediction.

One-half of twelve-thirteenths

The word "of" in a mathematical statement often indicates multiplication. In this problem, to calculate one-half *of* twelve-thirteenths, you should *multiply* one-half and twelve-thirteenths.

$$\frac{1}{2}\cdot\frac{12}{13}$$

To predict the value of the product, consider the literal translation of "one-half." Halving a quantity divides it by two. Hence, one-half of 12 would be 6, because $12 \div 2 = 6$. If you are given a quantity that consists of 12 "thirteenths," then half of that quantity is 6 "thirteenths," or $\frac{6}{13}$.

To verify the prediction, calculate the product of the fractions above, noting that 12 and 2 (a numerator-denominator pair) share the common factor 2, which can be eliminated from the product.

$$\frac{1}{2}\cdot\frac{12}{13}=\frac{1}{\cancel{2}\,1}\cdot\frac{\cancel{12}\,6}{13}$$
$$=\frac{1}{1}\cdot\frac{6}{13}$$
$$=\frac{1\cdot 6}{1\cdot 13}$$
$$=\frac{6}{13}$$

The product is equal to the prediction: $\frac{6}{13}$.

9.10 Simplify the expression: $\frac{21}{10}\cdot\frac{1}{60}\cdot\frac{6}{7}$.

Before you multiply the values in the numerators and divide by the product of the denominators, you should identify numerator-denominator pairs of numbers that share a common factor. There is no single, correct way to reduce the product, and all techniques, if performed correctly, will lead to the same answer.

For example, you may choose to divide the numerator value 6 and the denominator value 10 by common factor 2 as a first step. An equally valid approach is to divide the numerator value 6 and the denominator value 60 by common factor 6. This solution follows the latter approach—both will eventually result in the same product once the reduction process is complete.

$$\begin{aligned}\frac{21}{10}\cdot\frac{1}{60}\cdot\frac{6}{7} &= \frac{21}{10}\cdot\frac{1}{\cancel{60}\,10}\cdot\frac{\cancel{6}\,1}{7}\\ &= \frac{21}{10}\cdot\frac{1}{10}\cdot\frac{1}{7}\end{aligned}$$

Now reduce the numerator value 21 and the denominator value 7, which share common factor 7.

$$\begin{aligned}&= \frac{\cancel{21}\,3}{10}\cdot\frac{1}{10}\cdot\frac{1}{\cancel{7}\,1}\\ &= \frac{3}{10}\cdot\frac{1}{10}\cdot\frac{1}{1}\end{aligned}$$

The only numerator and denominator values that remain—other than 1—are 3 and 10; those numbers share no common factors, so you can calculate the product with the assurance that the result will be reduced to lowest terms.

$$\begin{aligned}&= \frac{3\cdot1\cdot1}{10\cdot10\cdot1}\\ &= \frac{3}{100}\end{aligned}$$

9.11 Simplify the expression: $\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdot\frac{5}{6}\cdot\frac{6}{7}$.

Five of the numerator values in this expression are equal to five of the denominator values: 2, 3, 4, 5, and 6. Except for the numerator value 1 and the denominator value 7, each of the other numbers in this expression is paired with a matching value on the other side of the fraction bar. Therefore, each of the numbers 2, 3, 4, 5, and 6 can be eliminated from the expression.

For example, consider the pair of 2s that appears in the numerator and denominator. They are equal, so their value is also the greatest common factor.

The GCF of 2 and 2 is 2. When you divide the 2s by 2, the result is $2 \div 2 = 1$. The same goes for all of the other equal pairs: The 3s become 1s, as do the 4s, 5s, and 6s.

$$
\begin{aligned}
\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdot\frac{5}{6}\cdot\frac{6}{7} &= \frac{1}{\cancel{2}\,1}\cdot\frac{\cancel{2}\,1}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdot\frac{5}{6}\cdot\frac{6}{7} \\
&= \frac{1}{1}\cdot\frac{1}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdot\frac{5}{6}\cdot\frac{6}{7}
\end{aligned}
$$

Similarly, the pair of 3s in the numerator and denominator shares a greatest common factor of 3.

$$
\begin{aligned}
&= \frac{1}{1}\cdot\frac{1}{\cancel{3}\,1}\cdot\frac{\cancel{3}\,1}{4}\cdot\frac{4}{5}\cdot\frac{5}{6}\cdot\frac{6}{7} \\
&= \frac{1}{1}\cdot\frac{1}{1}\cdot\frac{1}{4}\cdot\frac{4}{5}\cdot\frac{5}{6}\cdot\frac{6}{7}
\end{aligned}
$$

Continue to simplify the expression, eliminating the remaining pairs of numbers.

$$
\begin{aligned}
&= \frac{1}{1}\cdot\frac{1}{1}\cdot\frac{1}{\cancel{4}\,1}\cdot\frac{\cancel{4}\,1}{\cancel{5}\,1}\cdot\frac{\cancel{5}\,1}{\cancel{6}\,1}\cdot\frac{\cancel{6}\,1}{7} \\
&= \frac{1}{1}\cdot\frac{1}{1}\cdot\frac{1}{1}\cdot\frac{1}{1}\cdot\frac{1}{1}\cdot\frac{1}{7} \\
&= \frac{1}{7}
\end{aligned}
$$

At the outset of the solution, it was stated that only the values 1 and 7 did not have a corresponding, matching value opposite the fraction bar. Hence, they are the only values left once the product is reduced to lowest terms.

9.12 Simplify the expression: $\frac{8}{15}\cdot\frac{21}{34}\cdot\frac{25}{49}\cdot\frac{17}{20}$.

Reduce the product to lowest terms by dividing individual numerator-denominator pairs of numbers by common factors. As Problem 9.10 states, there may be alternate ways to choose these pairs of numbers, but all of them will lead to the same final answer. This solution begins by reducing the numerator value 21 and the denominator value 49, which share common factor 7.

The order in which you choose the pairs of numbers doesn't matter. You can wait to reduce the 21 and 49 pair until last if you want.

$$
\begin{aligned}
\frac{8}{15}\cdot\frac{21}{34}\cdot\frac{25}{49}\cdot\frac{17}{20} &= \frac{8}{15}\cdot\frac{\cancel{21}\,3}{34}\cdot\frac{25}{\cancel{49}\,7}\cdot\frac{17}{20} \\
&= \frac{8}{15}\cdot\frac{3}{34}\cdot\frac{25}{7}\cdot\frac{17}{20}
\end{aligned}
$$

The denominator value 34 is a multiple of the numerator value 17, so that pair of numbers has a greatest common factor of 17.

$$
\begin{aligned}
&= \frac{8}{15}\cdot\frac{3}{\cancel{34}\,2}\cdot\frac{25}{7}\cdot\frac{\cancel{17}\,1}{20} \\
&= \frac{8}{15}\cdot\frac{3}{2}\cdot\frac{25}{7}\cdot\frac{1}{20}
\end{aligned}
$$

The numerator value 25 and the denominator value 15 have common factor 5; similarly, the numerator value 8 and the denominator value 20 are both even, so they share common factor 2.

$$
\begin{aligned}
&= \frac{\cancel{8}\,4}{\cancel{15}\,3}\cdot\frac{3}{2}\cdot\frac{\cancel{25}\,5}{7}\cdot\frac{1}{\cancel{20}\,10}\\
&= \frac{4}{3}\cdot\frac{3}{2}\cdot\frac{5}{7}\cdot\frac{1}{10}
\end{aligned}
$$

In the aftermath of the previous steps, it is now possible to reduce further. The pair 4 and 2 shares common factor 2, the pair 3 and 3 have common factor 3, and the pair 5 and 10 share common factor 5.

$$
\begin{aligned}
&= \frac{\cancel{4}\,2}{\cancel{3}\,1}\cdot\frac{\cancel{3}\,1}{\cancel{2}\,1}\cdot\frac{\cancel{5}\,1}{7}\cdot\frac{1}{\cancel{10}\,2}\\
&= \frac{2}{1}\cdot\frac{1}{1}\cdot\frac{1}{7}\cdot\frac{1}{2}
\end{aligned}
$$

Finally, numerator value 2 and denominator value 2 share the obvious common factor 2.

$$
\begin{aligned}
&= \frac{\cancel{2}\,1}{1}\cdot\frac{1}{1}\cdot\frac{1}{7}\cdot\frac{1}{\cancel{2}\,1}\\
&= \frac{1}{1}\cdot\frac{1}{1}\cdot\frac{1}{7}\cdot\frac{1}{1}\\
&= \frac{1}{7}
\end{aligned}
$$

9.13 Simplify the expression: $\frac{3}{4}\cdot\frac{5}{8}-\frac{1}{12}$.

In this expression, you are directed to multiply and subtract fractions. The order of operations still applies to expressions containing fractions, so multiplication must be completed before subtraction.

$$
\begin{aligned}
\frac{3}{4}\cdot\frac{5}{8}-\frac{1}{12} &= \frac{3\cdot 5}{4\cdot 8}-\frac{1}{12}\\
&= \frac{15}{32}-\frac{1}{12}
\end{aligned}
$$

Only fractions with common denominators can be subtracted, as explained in Chapter 8. Thus, you need to calculate the least common denominator for the expression. The prime factorizations of the denominators are $12 = 2^2 \cdot 3$ and $32 = 2^5$. These factorizations contain the unique prime factors 2 and 3. The least common denominator is equal to the product of the prime factors, when each is raised to its highest power.

$$
\begin{aligned}
\text{LCD} &= 2^5\cdot 3\\
&= 32\cdot 3\\
&= 96
\end{aligned}
$$

Rewrite the expressions in terms of the least common denominator and combine the fractions.

$$\begin{aligned}\frac{15}{32}-\frac{1}{12}&=\frac{15\cdot 3}{32\cdot 3}-\frac{1\cdot 8}{12\cdot 8}\\&=\frac{45}{96}-\frac{8}{96}\\&=\frac{45-8}{96}\\&=\frac{37}{96}\end{aligned}$$

Dividing Proper Fractions

Keep, flip, change

9.14 Describe the Multiplicative Inverse Property, using $\frac{3}{5}$ as an example.

The reciprocal of a fraction is that fraction with its numerator and denominator reversed. The numerator of the original fraction is equal to the denominator of the reciprocal, and vice versa. Therefore, the reciprocal of $\frac{3}{5}$ is $\frac{5}{3}$.

According to the Multiplicative Inverse Property, any fraction multiplied by its reciprocal is equal to 1. Consider the product of $\frac{3}{5}$ and its reciprocal $\frac{5}{3}$.

$$\frac{3}{5}\cdot\frac{5}{3}$$

Because the reciprocal is defined in terms of the original fraction, any number in the numerator of one fraction will, by definition, appear in the denominator of the other fraction. This creates exactly two pairs of equal numbers that can be reduced using the method introduced in Problem 9.5. In this problem, both 3s can be divided by 3 and both 5s can be divided by 5.

$$\begin{aligned}\frac{3}{5}\cdot\frac{5}{3}&=\frac{\cancel{3}\,1}{\cancel{5}\,1}\cdot\frac{\cancel{5}\,1}{\cancel{3}\,1}\\&=\frac{1}{1}\cdot\frac{1}{1}\\&=1\end{aligned}$$

The product of any fraction and its reciprocal, like the product in this example, will always equal one.

9.15 Simplify the expression: $\frac{2}{9} \div \frac{5}{8}$.

By slightly modifying one of the fractions in the expression, you can change a quotient of fractions into a product. Specifically, change the divisor to its reciprocal and change the division symbol into a multiplication symbol.

$$\frac{2}{9} \div \frac{5}{8} = \frac{2}{9} \cdot \frac{8}{5}$$

Calculate the product using the technique described in Problems 9.1–9.13: Multiply the numerators and divide by the product of the denominators.

$$= \frac{2 \cdot 8}{9 \cdot 5}$$
$$= \frac{16}{45}$$

When you divide fractions, you Keep, Flip, and Change.

1. Keep the first fraction the way it is.
2. Flip the fraction you're dividing by.
3. Change division to multiplication.

9.16 Simplify the expression: $\frac{4}{15} \div \frac{7}{18}$.

Express the quotient as a product by replacing the divisor $\left(\frac{7}{18}\right)$ with its reciprocal $\left(\frac{18}{7}\right)$ and replacing the division symbol with a multiplication symbol.

$$\frac{4}{15} \div \frac{7}{18} = \frac{4}{15} \cdot \frac{18}{7}$$

Notice that the numerator value 18 and the denominator value 15 share common factor 3. Reduce the product, dividing the pair of numbers by the common factor (the process first described in Problem 9.5).

You can only "cancel out" like this when you're multiplying. You can't do it while the ÷ symbol is still in there.

$$= \frac{4}{\cancel{15}\,5} \cdot \frac{\cancel{18}\,6}{7}$$
$$= \frac{4}{5} \cdot \frac{6}{7}$$

Calculate the product of the fractions.

$$= \frac{4 \cdot 6}{5 \cdot 7}$$
$$= \frac{24}{35}$$

9.17 Simplify the expression: $\frac{1}{12} \div 3$.

To express an integer as a fraction, divide it by 1: $3 = \frac{3}{1}$. Rewrite the expression as the quotient of two fractions.

$$\frac{1}{12} \div 3 = \frac{1}{12} \div \frac{3}{1}$$

Convert the division problem into a multiplication problem by changing the divisor to its reciprocal.

$$=\frac{1}{12}\cdot\frac{1}{3}$$

Calculate the product.

$$=\frac{1\cdot 1}{12\cdot 3}$$
$$=\frac{1}{36}$$

If you ask "How many one-halves fit into 10?" the answer is not 5. As you'll see, the answer is 20. You can fit twice as many one-halves into 10 than you can fit ones into 10, and $2(10) = 20$.

9.18 Simplify the expression: $10 \div \frac{1}{2}$.

This problem encompasses a very common student error. You are directed to divide 10 by $\frac{1}{2}$, which is *not* the same as calculating half of 10. Make sure to employ the division technique described in Problem 9.15 instead of applying (in this case) faulty logic to produce the inaccurate quotient 5.

Replace $\frac{1}{2}$ with its reciprocal $\frac{2}{1}$; change the division symbol to a multiplication symbol, and calculate the product.

$$10 \div \frac{1}{2} = \frac{10}{1} \div \frac{1}{2}$$
$$= \frac{10}{1}\cdot\frac{2}{1}$$
$$= \frac{10\cdot 2}{1\cdot 1}$$
$$= \frac{20}{1}$$
$$= 20$$

9.19 Simplify the expression: $\frac{7}{9} \div \frac{14}{15}$.

Rewrite the expression, replacing the divisor with its reciprocal and changing the quotient into a product.

$$\frac{7}{9} \div \frac{14}{15} = \frac{7}{9}\cdot\frac{15}{14}$$

The numerator value 7 and the denominator value 14 share common factor 7; similarly, 9 and 15 share common factor 3. Reduce the product to lowest terms.

$$= \frac{\cancel{7}\,1}{\cancel{9}\,3} \cdot \frac{\cancel{15}\,5}{\cancel{14}\,2}$$
$$= \frac{1}{3} \cdot \frac{5}{2}$$
$$= \frac{1 \cdot 5}{3 \cdot 2}$$
$$= \frac{5}{6}$$

9.20 Simplify the expression: $\frac{24}{35} \div \frac{16}{25} + \frac{4}{21}$.

The order of operations dictates that division precedes addition. Rewrite the quotient as a product and reduce it to lowest terms. Make no changes to the fraction $\frac{4}{21}$ until the quotient-turned-product is simplified.

$$\frac{24}{35} \div \frac{16}{25} + \frac{4}{21} = \frac{24}{35} \cdot \frac{25}{16} + \frac{4}{21}$$

Identify the numerator-denominator pairs that share a common factor: 24 and 16 share greatest common factor 8; 25 and 35 share common factor 5.

$$= \frac{\cancel{24}\,3}{\cancel{35}\,7} \cdot \frac{\cancel{25}\,5}{\cancel{16}\,2} + \frac{4}{21}$$
$$= \frac{3}{7} \cdot \frac{5}{2} + \frac{4}{21}$$
$$= \frac{3 \cdot 5}{7 \cdot 2} + \frac{4}{21}$$
$$= \frac{15}{14} + \frac{4}{21}$$

Rewrite the expression in terms of the least common denominator, 42, and combine the fractions.

$$= \frac{15 \cdot 3}{14 \cdot 3} + \frac{4 \cdot 2}{21 \cdot 2}$$
$$= \frac{45}{42} + \frac{8}{42}$$
$$= \frac{45 + 8}{42}$$
$$= \frac{53}{42}$$

Unless otherwise directed by your instructor, improper fractions should not be converted to mixed numbers or decimals.

9.21 Simplify the expression: $\frac{16}{19}\cdot\frac{3}{7}\div\frac{32}{35}$.

The order of operations dictates that multiplication and division should occur from left to right. Thus, you should multiply $\frac{16}{19}\cdot\frac{3}{7}$ before you divide by $\frac{32}{35}$.

$$\frac{16}{19}\cdot\frac{3}{7}\div\frac{32}{35}=\frac{16\cdot 3}{19\cdot 7}\div\frac{32}{35}$$

Rather than multiply the values in the numerator and denominator ($16\cdot 3$ and $19\cdot 7$), leave them as products. This will assist in the identification of common factors in the next step. Express the quotient as a product.

$$=\frac{16\cdot 3}{19\cdot 7}\cdot\frac{35}{32}$$

Reduce the fraction: 16 and 32 share common factor 16; 35 and 7 share common factor 7.

$$\begin{aligned}&=\frac{\cancel{16}\,1\cdot 3}{19\cdot\cancel{7}\,1}\cdot\frac{\cancel{35}\,5}{\cancel{32}\,2}\\&=\frac{1\cdot 3}{19\cdot 1}\cdot\frac{5}{2}\\&=\frac{1\cdot 3\cdot 5}{19\cdot 1\cdot 2}\\&=\frac{15}{38}\end{aligned}$$

9.22 Simplify the expression: $\frac{3}{5}\div\frac{8}{9}\div\frac{21}{10}$.

This expression contains two quotients. Calculate them one at a time, in order from left to right. Begin by expressing $\frac{3}{5}\div\frac{8}{9}$ as a product.

$$\begin{aligned}\frac{3}{5}\div\frac{8}{9}\div\frac{21}{10}&=\frac{3}{5}\cdot\frac{9}{8}\div\frac{21}{10}\\&=\frac{3\cdot 9}{5\cdot 8}\div\frac{21}{10}\end{aligned}$$

Like Problem 9.21, the left fraction contains a product that shouldn't be simplified until the remaining quotient is rewritten as a product in the next step.

It's not that big a deal. You can go ahead and multiply the first two fractions to get $\frac{27}{40}$. You'll just need to recognize that 10 and 40 have a common factor, instead of 10 and 5; 27 and 21 have a common factor, instead of 3 and 21.

$$=\frac{3\cdot 9}{5\cdot 8}\cdot\frac{10}{21}$$

Reduce the product: 3 and 21 share common factor 3; 5 and 10 share common factor 5.

$$
\begin{aligned}
&= \frac{\cancel{3}\,1 \cdot 9}{\cancel{5}\,1 \cdot 8} \cdot \frac{\cancel{10}\,2}{\cancel{21}\,7} \\
&= \frac{1 \cdot 9}{1 \cdot 8} \cdot \frac{2}{7} \\
&= \frac{1 \cdot 9 \cdot 2}{1 \cdot 8 \cdot 7}
\end{aligned}
$$

The fraction is not yet reduced to lowest terms—2 and 8 share common factor 2.

$$
\begin{aligned}
&= \frac{1 \cdot 9 \cdot \cancel{2}\,1}{1 \cdot \cancel{8}\,4 \cdot 7} \\
&= \frac{1 \cdot 9 \cdot 1}{1 \cdot 4 \cdot 7} \\
&= \frac{9}{28}
\end{aligned}
$$

9.23 Simplify the expression: $\frac{7}{12} \div \frac{3}{4} \div 9$.

Like Problem 9.22, this expression requires you to divide twice. Begin by calculating the left quotient.

$$\frac{7}{12} \div \frac{3}{4} \div 9 = \frac{7}{12} \cdot \frac{4}{3} \div 9$$

Reduce the product to lowest terms—there is no need to wait until you divide by 9 to do so.

$$
\begin{aligned}
&= \frac{7}{\cancel{12}\,3} \cdot \frac{\cancel{4}\,1}{3} \div 9 \\
&= \frac{7}{3} \cdot \frac{1}{3} \div 9 \\
&= \frac{7 \cdot 1}{3 \cdot 3} \div 9 \\
&= \frac{7}{9} \div 9
\end{aligned}
$$

Divide 4 and 12 by their greatest common factor: 4.

Express the integer 9 as the fraction $\frac{9}{1}$ and calculate the final quotient.

$$
\begin{aligned}
&= \frac{7}{9} \div \frac{9}{1} \\
&= \frac{7}{9} \cdot \frac{1}{9} \\
&= \frac{7 \cdot 1}{9 \cdot 9} \\
&= \frac{7}{81}
\end{aligned}
$$

Improper Fractions and Mixed Numbers

Up for some conversion?

9.24 Express the product as an improper fraction: $\frac{5}{3}\cdot\frac{11}{8}$.

The procedure used to multiply proper fractions (explored in Problems 9.1–9.13) is also used to multiply improper fractions. The comparative values of the numerator and denominator are irrelevant—the product is still equal to the product of the numerators divided by the product of the denominators.

$$\begin{aligned}\frac{5}{3}\cdot\frac{11}{8}&=\frac{5\cdot 11}{3\cdot 8}\\&=\frac{55}{24}\end{aligned}$$

Unless otherwise instructed, you should not convert an improper fraction solution into a mixed number or a decimal.

9.25 Simplify the expression: $\frac{24}{13}\div\frac{9}{5}$.

Rewrite the quotient as a product and reduce it to lowest terms.

Divide 24 and 9 by the GCF: 3.

$$\begin{aligned}\frac{24}{13}\div\frac{9}{5}&=\frac{24}{13}\cdot\frac{5}{9}\\&=\frac{\cancel{24}\,8}{13}\cdot\frac{5}{\cancel{9}\,3}\\&=\frac{8}{13}\cdot\frac{5}{3}\\&=\frac{8\cdot 5}{13\cdot 3}\\&=\frac{40}{39}\end{aligned}$$

9.26 Simplify the expression: $\frac{121}{2}\div\frac{14}{3}\div\frac{55}{42}$.

Rewrite the quotients as products, one at a time, in order from left to right. As first discussed in Problem 9.21, when three or more fractions are present, you should avoid multiplying until you have reduced the product to lowest terms.

$$\begin{aligned}\frac{121}{2}\div\frac{14}{3}\div\frac{55}{42}&=\frac{121}{2}\cdot\frac{3}{14}\div\frac{55}{42}\\&=\frac{121\cdot 3}{2\cdot 14}\div\frac{55}{42}\\&=\frac{121\cdot 3}{2\cdot 14}\cdot\frac{42}{55}\end{aligned}$$

Numerator value 121 and denominator value 55 share greatest common factor 11; similarly, 42 and 14 share greatest common factor 14.

$$
\begin{aligned}
&= \frac{\cancel{121}\,11\cdot 3}{2\cdot \cancel{14}\,1}\cdot\frac{\cancel{42}\,3}{\cancel{55}\,5}\\
&= \frac{11\cdot 3}{2\cdot 1}\cdot\frac{3}{5}\\
&= \frac{33}{2}\cdot\frac{3}{5}\\
&= \frac{33\cdot 3}{2\cdot 5}\\
&= \frac{99}{10}
\end{aligned}
$$

9.27 Determine whether the following statement is true or false and explain your answer.

$$\left(3\frac{1}{2}\right)\left(4\frac{3}{5}\right)=3\cdot\frac{1}{2}\cdot 4\cdot\frac{3}{5}$$

According to Problem 8.33, you can add mixed numbers by adding the integer parts and proper fractions individually, and then combining the results. This problem asks whether that method applies to multiplication as well. In other words, it asks whether you can multiply the individual pieces of two mixed numbers to calculate their product. To determine whether or not the statement is true, you should evaluate the expressions on both sides of the equal sign and then compare the values.

The product on the right side of the equation is relatively straightforward. Convert the integers to fractions, reduce the product to lowest terms, and multiply.

$$
\begin{aligned}
3\cdot\frac{1}{2}\cdot 4\cdot\frac{3}{5} &= \frac{3}{1}\cdot\frac{1}{2}\cdot\frac{4}{1}\cdot\frac{3}{5}\\
&= \frac{3}{1}\cdot\frac{1}{\cancel{2}\,1}\cdot\frac{\cancel{4}\,2}{1}\cdot\frac{3}{5}\\
&= \frac{3\cdot 1\cdot 2\cdot 3}{1\cdot 1\cdot 1\cdot 5}\\
&= \frac{18}{5}
\end{aligned}
$$

In other words, turn them into improper fractions and then multiply them like in Problems 9.24–9.26.

The proper way to calculate the product of two mixed numbers—and thus to evaluate the left side of the original equation—is to convert the mixed numbers to improper fractions and then multiply.

$$\begin{aligned}\left(3\frac{1}{2}\right)\left(4\frac{3}{5}\right)&=\left(\frac{2\cdot 3+1}{2}\right)\left(\frac{5\cdot 4+3}{5}\right)\\&=\left(\frac{6+1}{2}\right)\left(\frac{20+3}{5}\right)\\&=\frac{7}{2}\cdot\frac{23}{5}\\&=\frac{7\cdot 23}{2\cdot 5}\\&=\frac{161}{10}\end{aligned}$$

Because $\frac{18}{5}\neq\frac{161}{10}$, you conclude that the original statement is false. The product of two mixed numbers is not equal to the product of the component integer and proper fraction parts. Rather, the product is best calculated by converting the mixed numbers to improper fractions before multiplying the values.

Note: In Problems 9.28–9.29, you test the validity of a shortcut for dividing mixed numbers.

9.28 Simplify the expression: $\left(1\frac{3}{4}\right)\div\left(7\frac{1}{2}\right)$.

Convert the mixed numbers to improper fractions and rewrite the quotient as a product.

$$\begin{aligned}\left(1\frac{3}{4}\right)\div\left(7\frac{1}{2}\right)&=\left(\frac{4\cdot 1+3}{4}\right)\div\left(\frac{2\cdot 7+1}{2}\right)\\&=\frac{7}{4}\div\frac{15}{2}\\&=\frac{7}{4}\cdot\frac{2}{15}\end{aligned}$$

Reduce the product to lowest terms and multiply the fractions.

$$\begin{aligned}&=\frac{7}{\cancel{4}\,2}\cdot\frac{\cancel{2}\,1}{15}\\&=\frac{7}{2}\cdot\frac{1}{15}\\&=\frac{7}{30}\end{aligned}$$

Note: In Problems 9.28–9.29, you test the validity of a shortcut for dividing mixed numbers.

9.29 Determine whether or not the following statement is true.

$$\left(1\frac{3}{4}\right)\div\left(7\frac{1}{2}\right)=(1\div 7)+\left(\frac{3}{4}\div\frac{1}{2}\right)$$

Evaluate the right side of the equation, expressing $1 \div 7$ as a fraction and calculating the quotient $\frac{3}{4}\div\frac{1}{2}$.

$$\begin{aligned}(1\div 7)+\left(\frac{3}{4}\div\frac{1}{2}\right)&=\frac{1}{7}+\left(\frac{3}{4}\div\frac{1}{2}\right)\\&=\frac{1}{7}+\left(\frac{3}{4}\cdot\frac{2}{1}\right)\\&=\frac{1}{7}+\left(\frac{3}{\cancel{4}\,2}\cdot\frac{\cancel{2}\,1}{1}\right)\\&=\frac{1}{7}+\left(\frac{3}{2}\cdot\frac{1}{1}\right)\\&=\frac{1}{7}+\frac{3}{2}\end{aligned}$$

Write the fractions in terms of the least common denominator (14) and simplify the expression.

$$\begin{aligned}&=\frac{1\cdot 2}{7\cdot 2}+\frac{3\cdot 7}{2\cdot 7}\\&=\frac{2}{14}+\frac{21}{14}\\&=\frac{23}{14}\end{aligned}$$

According to Problem 9.28, the quotient of the mixed numbers should be $\frac{7}{30}$, so the statement is false and the shortcut presented in this problem is invalid. You cannot divide mixed numbers by dividing the whole numbers and proper fractions individually and then adding those quotients.

$$\left(1\frac{3}{4}\right)\div\left(7\frac{1}{2}\right)\neq(1\div 7)+\left(\frac{3}{4}\div\frac{1}{2}\right)$$

$$\frac{7}{30}\neq\frac{23}{14}$$

9.30 Simplify the expression: $\left(6\frac{4}{9}\right)\div\left(-2\frac{2}{3}\right)$.

Convert the mixed numbers to improper fractions and calculate the quotient.

$$\begin{aligned}\left(6\frac{4}{9}\right)\div\left(-2\frac{2}{3}\right)&=\left(\frac{9\cdot 6+4}{9}\right)\div\left(-\frac{3\cdot 2+2}{3}\right)\\&=\frac{58}{9}\div\left(-\frac{8}{3}\right)\\&=\frac{58}{9}\left(-\frac{3}{8}\right)\end{aligned}$$

A positive fraction times a negative fraction equals a negative fraction, just like a positive integer times a negative integer equals a negative integer.

Reduce the product to lowest terms and multiply the fractions.

$$\begin{aligned}&=\frac{\cancel{58}\,29}{\cancel{9}\,3}\left(-\frac{\cancel{3}\,1}{\cancel{8}\,4}\right)\\&=\frac{29}{3}\left(-\frac{1}{4}\right)\\&=-\frac{29}{12}\end{aligned}$$

Complex Fractions

Fractions within fractions within a crunchy candy coating

9.31 Identify the defining characteristic of a complex fraction and describe the method used to simplify complex fractions.

A complex fraction has a numerator or a denominator (or both a numerator and a denominator) that is, itself, a fraction. Follow the steps below to simplify a complex fraction:

1. Write the complex fraction as a quotient, the numerator divided by the denominator.
2. Use the technique described in Problem 9.15 to convert the quotient to a product.
3. Reduce the product to lowest terms.
4. Multiply the fractions.

9.32 Simplify the complex fraction: $\frac{2/5}{4/5}$.

Follow the steps outlined in Problem 9.31, beginning by expressing the complex fraction as a quotient of two fractions.

$$\frac{2/5}{4/5}=\frac{2}{5}\div\frac{4}{5}$$

Rewrite the quotient as a product.

$$=\frac{2}{5}\cdot\frac{5}{4}$$

The value 5 appears in the numerator and the denominator; reduce the product by eliminating the common factor.

$$\begin{aligned}&=\frac{2}{\cancel{5}1}\cdot\frac{\cancel{5}1}{4}\\&=\frac{2}{1}\cdot\frac{1}{4}\\&=\frac{2}{4}\end{aligned}$$

This fraction is not yet reduced to lowest terms, but note that its numerator and denominator are reminiscent of the original fraction, $\frac{2/5}{4/5}$. The numerator and denominator of the answer match the numerators in the complex fraction: 2 appears above the fraction bar and 4 appears below it. This result is not unique to this problem. Reduce the fraction to lowest terms.

SHORTCUT ALERT! If the denominators in a complex fraction match, just cross them out. In this case, 5 is the denominator of $\frac{2}{5}$ and $\frac{4}{5}$, so erase the 5's and you're left with $\frac{2}{4}$.

$$\begin{aligned}&=\frac{2\div 2}{4\div 2}\\&=\frac{1}{2}\end{aligned}$$

9.33 Simplify the complex fraction: $\frac{1/3}{7/3}$.

Like Problem 9.32, the denominators in this complex fraction are equal. Thus, the result should be the quotient of the numerators: $\frac{1/3}{7/3}=\frac{1}{7}$. Verify this result using the procedure outlined in Problem 9.31.

This is the shortcut I'm talking about in Problem 9.32.

$$\begin{aligned}\frac{1/3}{7/3}&=\frac{1}{3}\div\frac{7}{3}\\&=\frac{1}{3}\cdot\frac{3}{7}\\&=\frac{1}{\cancel{3}1}\cdot\frac{\cancel{3}1}{7}\\&=\frac{1}{1}\cdot\frac{1}{7}\\&=\frac{1}{7}\end{aligned}$$

9.34 Simplify the complex fraction: $\dfrac{5/6}{5/9}$.

Express the complex fraction as a quotient of the numerator and denominator.

$$\frac{5/6}{5/9} = \frac{5}{6} \div \frac{5}{9}$$
$$= \frac{5}{6} \cdot \frac{9}{5}$$

Reduce the product to lowest terms and multiply the fractions.

$$= \frac{\cancel{5}\,1}{\cancel{6}\,2} \cdot \frac{\cancel{9}\,3}{\cancel{5}\,1}$$
$$= \frac{1}{2} \cdot \frac{3}{1}$$
$$= \frac{3}{2}$$

The numerators of the original complex fraction are equal. Here's a shortcut: Keep only the denominators $\left(\frac{6}{9}\right)$ and then take the reciprocal $\left(\frac{9}{6}\right)$. Of course, you'll have to reduce, too.

9.35 Simplify the complex fraction: $\dfrac{5}{7/10}$.

Express the numerator as a fraction and then rewrite the complex fraction as a quotient.

$$\frac{5}{7/10} = \frac{5/1}{7/10}$$
$$= \frac{5}{1} \div \frac{7}{10}$$
$$= \frac{5}{1} \cdot \frac{10}{7}$$

Calculate the product.

$$= \frac{5 \cdot 10}{1 \cdot 7}$$
$$= \frac{50}{7}$$

9.36 Simplify the complex fraction: $\dfrac{22/13}{4}$.

Like Problem 9.35, part of this complex fraction should first be converted to a fraction itself before calculating the quotient: $4 = \frac{4}{1}$.

$$\frac{22/13}{4} = \frac{22/13}{4/1}$$
$$= \frac{22}{13} \div \frac{4}{1}$$
$$= \frac{22}{13} \cdot \frac{1}{4}$$

Reduce the product and multiply the fractions.

$$\begin{aligned}
&= \frac{\cancel{22}\,11}{13} \cdot \frac{1}{\cancel{4}\,2} \\
&= \frac{11}{13} \cdot \frac{1}{2} \\
&= \frac{11}{26}
\end{aligned}$$

9.37 Simplify the complex fraction: $\dfrac{24/65}{8/39}$.

Express the complex fraction as the quotient of the numerator and denominator.

$$\begin{aligned}
\frac{24/65}{8/39} &= \frac{24}{65} \div \frac{8}{39} \\
&= \frac{24}{65} \cdot \frac{39}{8}
\end{aligned}$$

Numerator value 24 and denominator value 8 share greatest common factor 8; similarly, 39 and 65 share greatest common factor 13.

$$\begin{aligned}
&= \frac{\cancel{24}\,3}{\cancel{65}\,5} \cdot \frac{\cancel{39}\,3}{\cancel{8}\,1} \\
&= \frac{3}{5} \cdot \frac{3}{1} \\
&= \frac{9}{5}
\end{aligned}$$

9.38 Simplify the expression: $\dfrac{9/2}{15/7} + \dfrac{3}{8}$.

Before you can address the sum, you must first simplify the complex fraction.

$$\begin{aligned}
\frac{9/2}{15/7} + \frac{3}{8} &= \left(\frac{9}{2} \div \frac{15}{7}\right) + \frac{3}{8} \\
&= \left(\frac{9}{2} \cdot \frac{7}{15}\right) + \frac{3}{8} \\
&= \left(\frac{\cancel{9}\,3}{2} \cdot \frac{7}{\cancel{15}\,5}\right) + \frac{3}{8} \\
&= \left(\frac{3 \cdot 7}{2 \cdot 5}\right) + \frac{3}{8} \\
&= \frac{21}{10} + \frac{3}{8}
\end{aligned}$$

If you need help, check out Problem 8.10. It contains fractions with the same denominators. (Which are, ironically, 8 and 10.) Weird.

Rewrite the expression in terms of the least common denominator and add the fractions.

$$\begin{aligned}&=\frac{21\cdot 4}{10\cdot 4}+\frac{3\cdot 5}{8\cdot 5}\\&=\frac{84}{40}+\frac{15}{40}\\&=\frac{99}{40}\end{aligned}$$

9.39 Simplify the complex fraction: $\dfrac{\frac{1/4}{5/12}}{\frac{27}{25}}$.

Begin by simplifying the complex fraction in the numerator.

$$\begin{aligned}\frac{1/4}{5/12}&=\frac{1}{4}\div\frac{5}{12}\\&=\frac{1}{4}\cdot\frac{12}{5}\\&=\frac{1}{\cancel{4}\,1}\cdot\frac{\cancel{12}\,3}{5}\\&=\frac{1\cdot 3}{1\cdot 5}\\&=\frac{3}{5}\end{aligned}$$

Substitute this fraction into the original expression and simplify the resulting complex fraction.

$$\begin{aligned}\frac{\frac{1/4}{5/12}}{\frac{27}{25}}&=\frac{3/5}{27/25}\\&=\frac{3}{5}\div\frac{27}{25}\\&=\frac{3}{5}\cdot\frac{25}{27}\\&=\frac{\cancel{3}\,1}{\cancel{5}\,1}\cdot\frac{\cancel{25}\,5}{\cancel{27}\,9}\\&=\frac{1\cdot 5}{1\cdot 9}\\&=\frac{5}{9}\end{aligned}$$

9.40 Simplify the complex fraction: $\dfrac{\frac{2/9}{5/14}}{\frac{7/8}{6/11}}$.

Simplify the complex fractions in the numerator and denominator independently.

$$\frac{2/9}{5/14}=\frac{2}{9}\div\frac{5}{14} \qquad \frac{7/8}{6/11}=\frac{7}{8}\div\frac{6}{11}$$
$$=\frac{2}{9}\cdot\frac{14}{5} \qquad =\frac{7}{8}\cdot\frac{11}{6}$$
$$=\frac{28}{45} \qquad =\frac{77}{48}$$

Replace the complex fractions in the original expression with the fractions calculated above.

$$\frac{\frac{2/9}{5/14}}{\frac{7/8}{6/11}}=\frac{28/45}{77/48}$$
$$=\frac{28}{45}\div\frac{77}{48}$$
$$=\frac{28}{45}\cdot\frac{48}{77}$$

Note that numerator value 28 and denominator value 77 share greatest common factor 7. Similarly, 48 and 45 share greatest common factor 3.

$$=\frac{\cancel{28}\,4}{\cancel{45}\,15}\cdot\frac{\cancel{48}\,16}{\cancel{77}\,11}$$
$$=\frac{4\cdot 16}{15\cdot 11}$$
$$=\frac{64}{165}$$

Chapter 10
DECIMALS

Time for a "point"ed discussion

The preceding three chapters have discussed, in some depth, the concept of fractions. Fractions are, by definition, a means by which to express a quotient, but they are also commonly used to represent values that aren't whole numbers. (They aren't restricted from such representations, of course, but the set of rational numbers dwarfs the set of whole numbers.)

Decimals present another method by which to express non-whole number values by extending the place values discussed in Chapter 1 across a punctuation mark called a "decimal point." Though it resembles a period, a decimal point is placed in the *middle* of a decimal, rather than at the end of a numerical sentence. This chapter identifies the additional place values introduced by decimal notation and explains how to apply the four basic binary operations (addition, subtraction, multiplication, and division) to decimals.

When you see the fraction $\frac{1}{2}$, you might think "point five," as in "one half is equal to 0.5." However, knowing that a decimal point looks like a point (or a "dot") inside a number doesn't help you add, subtract, multiply, or divide them. Luckily, those four operations work almost the same way as they do with regular, run-of-the-mill whole numbers. Each operation has its own little extra twist you need to remember, but they all build on the procedures that are explained in detail in Chapter 2.

Before you can even think about working with decimals, you need to understand their place values, so the first two sections of the chapter start there.

Place Values

Ten, hundred, and thousand + "ths"

10.1 List the place values for each digit in the number 2.6 and use that information to identify the number.

As explained in Chapter 1, the rightmost digit of a whole number—a number that does not contain a decimal portion—is the ones place. The digit 2 appears in the ones place of the number 2.6, as it is the digit immediately to the left of the decimal point.

You could also read it "2 point 6," but that's sort of informal. Use the word "and" instead of "point."

The first decimal digit of a number appears directly on the opposite side of the decimal point, and it occupies the "tenths place." Thus, 6 is in the tenths place. The number "2.6" is read "two and six tenths." Note that the word "and" is used to indicate the position of the decimal point, separating the ones place from the tenths place.

10.2 List the place values for each digit in the number 15.37 and use that information to identify the number.

The digits 1 and 5 appear left of the decimal point and occupy (in order from right to left) the ones and tens place.

- 5 is in the ones place
- 1 is in the tens place

You name whole digits from right to left and decimal digits from left to right.

The digits 3 and 7 appear right of the decimal point and occupy (in order *from left to right*), the tenths and hundredths place.

- 3 is in the tenths place
- 7 is in the hundredths place

The number 15.37 is read "fifteen and thirty-seven hundredths." The key to identifying decimals is the rightmost decimal place value; only that decimal place is explicitly named at the end of the number. In this case, there are two digits right of the decimal point, so the number ends in "hundredths."

10.3 List the place values for each digit in the number 406.912 and use that information to identify the number.

This number contains three digits right of the decimal point, so it ends in the "thousandths" place.

- 4 is in the hundreds place
- 0 is in the tens place
- 6 is in the ones place
- 9 is in the tenths place
- 1 is in the hundredths place
- 2 is in the thousandths place

The number 406.912 is read "four hundred six and nine hundred twelve thousandths."

10.4 Express the number as a decimal: nine and four tenths.

The word "and" separates the whole portion of the number from the decimal portion; the whole portion appears before the word, and the decimal portion appears immediately after it. Therefore, the whole portion of this number is nine and the decimal portion is four tenths.

The decimal is presented as "tenths," which means one digit will appear to the right of the decimal point, in this case the digit 4. Therefore, the numerical representation of nine and four tenths is 9.4.

10.5 Express the number as a decimal: one hundred fifty-six and seventy-nine hundredths.

The whole portion of the number is 156. If you need help with this step, consult the problems at the beginning of Chapter 1, which discuss place values of whole numbers. The key to deciphering the decimal portion of a number is the final word in the phrase, which ends in "ths." Here, the word "hundredths" indicates that exactly two digits should appear right of the decimal point. Consider the terms below and the corresponding number of digits that should appear right of the decimal point for each.

Key Word	*Decimal Digits*
tenths	1
hundredths	2
thousandths	3
ten thousandths	4
hundred thousandths	5
millionths	6
billionths	9

This isn't true in the next problem. The decimal digits and the expected digits don't match.

The decimal representation of one hundred fifty-six and seventy-nine hundredths is 156.79. Notice that the decimal portion, "seventy-nine," is a two-digit number and the word "hundredths" indicates that exactly two digits should appear right of the decimal point.

10.6 Express the number as a decimal: eighty-two and eleven thousandths.

The whole portion of this number is 82. However, the decimal portion, 11, is a two-digit number and the word "thousandths" indicates that three decimal digits should be included. In these instances, "place holder" zeroes should be placed *before* the non-zero digits, immediately after the decimal point.

For example, in this problem, three digits are needed after the decimal, but 11 only occupies two of those places. Therefore, a single zero should be placed before 11, after the decimal point. The decimal representation of eighty-two and eleven thousandths is 82.011.

Zeroes placed at the end of a decimal have no effect on the decimal's value. In other words, there's no reason to write "3.9600000" instead of "3.96" unless you're demonstrating how exact a measurement is.

10.7 Express the number as a decimal: Four and one ten thousandth.

The whole portion of this number is 4 and the decimal portion contains a single explicitly stated digit: 1. However, according to the chart presented in Problem 10.5, the words "ten thousandth" indicate that four digits must appear after the decimal point. Therefore, three zeroes should precede the digit 1 to ensure that the required number of digits appears in the decimal portion. The decimal representation of four and one ten thousandth is 4.0001.

Rounding

What's the nearest "th"?

10.8 Round 94.63 to the nearest tenth.

Review the technique described in Problem 1.11 that is used to round whole numbers. It is based upon the digit immediately to the right of the place to which you are rounding. If that digit is less than 5, change it (and any digits right of it) to 0. If the digit is 5 or greater, you still change that digit (and the digits to its right) to 0, but this time you add 1 to the place to which you are rounding.

Rounding decimals works similarly. You still base any decision about rounding upon the digit immediately to the right of the place to which you are rounding. In this case, you are asked to round to the nearest tenth, so you consider the hundredths digit 3, which is immediately to the right of the tenths digit 6. Because $3 < 5$, you change the hundredths digit to 0 and leave the tenths digit alone. The number 94.63, rounded to the nearest tenth, is 94.6.

10.9 Round 8.1794 to the nearest hundredth.

The hundredths digit of 8.1794 is 7, so you must decide whether to leave the digit alone (resulting in a rounded value of 8.17) or to add 1 to the digit (resulting in 8.18). Consider the digit immediately to the right of 7, the thousandths digit 9.

Always use 5 as the deciding value for rounding.
- Less than 5? Round down.
- Greater than or equal to 5? Round up.

Because $9 \geq 5$, you should add 1 to the hundredths place and change all digits that are right of the hundredths place to 0s: 8.1800. There is no need to list 0 digits that appear at the end of a decimal, so your answer is more correct if they are omitted. The number 8.1794, rounded to the nearest hundredth, is 8.18.

10.10 Round 3.141592 to the nearest (a) tenth, (b) thousandth, and (c) whole number.

The tenths digit of 3.141592 is 1. Note that the digit immediately to its right—the hundredths digit 4—is less than 5. Therefore, the tenths digit is unchanged and all decimal digits beginning with the hundredths place to the end of the number are omitted. The number 3.141592, rounded to the nearest tenth, is 3.1.

The thousandths digit of 3.141592 is also 1; consider the digit to its right, the ten thousandths digit 5. Because this digit is greater than *or equal to* 5, add one to the thousandths digit and omit all digits to the right of the newly changed place value. The number 3.141592, rounded to the nearest thousandth, is 3.142.

When a problem asks you to round to the nearest whole number, it is asking you to round to the ones place. In this case, the problem asks you to decide whether 3.141592 is closer to 3 or 4. To decide, consider the digit to the right of the ones place—the tenths place, in this case 1. Because $1 < 5$, you leave the ones digit unchanged and omit the entire decimal portion of the number. The number 3.141592, rounded to the nearest whole number, is 3.

You are asked to round to the nearest WHOLE number, so it makes sense to leave off the decimals.

10.11 Round 821.0007 to the nearest thousandth.

The thousandths digit of 821.0007 is the third digit right of the decimal point—in this case, the third 0 as you read from left to right. The digit immediately to its right, the ten thousandths digit, is 7. Because $7 \geq 5$, you should add 1 to the thousandths digit $(0 + 1 = 1)$ and truncate the numeral at the thousandths digit. The number 821.0007, rounded to the nearest thousandth, is 821.001.

"Truncate" means end the decimal by chopping off the rest of the decimal digits.

10.12 Round 483.52967 to the nearest hundred.

Make sure to read the directions to this problem carefully. You are asked to round to the nearest *hundred*, not to the nearest *hundredth*. The hundreds place is the third digit left of the decimal point, in this case the digit 4. This problem is more like Problem 1.13, which also asks you to round to the hundreds place, than Problem 10.9, which involves rounding to the hundredths place. In fact, the decimal portion of this number does not influence the answer at all.

To round 483.52967, consider the digit immediately to the right of 4, the tens digit 8. Because $8 \geq 5$, you should add 1 to the hundreds digit and change all digits right of the hundreds digit to 0s.

$$483.52967 \approx 500.00000$$

In other words, 500.000009 is fine, but 500.00000 is not.

Recall that zeroes after a decimal are unnecessary unless a non-zero number follows them. In this case, the only non-zero digit is 5 and it appears before the decimal point. Therefore, your final answer should contain no decimal portion. The number 483.52967, rounded to the nearest hundred, is 500.

10.13 Round 4.8399992 to the nearest hundred thousandth.

Identify the place values of the decimal digits.

- 8 is in the tenths place
- 3 is in the hundredths place
- 9 is in the thousandths place
- 9 is in the ten thousandths place
- 9 is in the hundred thousandths place
- 9 is in the millionths place
- 2 is in the ten millionths place

See Problem 1.14 if you need help with this.

Consider the millionths digit 9, which is immediately to the right of the hundred thousandths digit to which you are rounding. Because $9 \geq 5$, you should add 1 to the hundred thousandths digit. However, that produces a two digit number: $9 + 1 = 10$. Thus, you should record a 0 for the hundred thousandths digit and, in turn, add 1 to the digit on its left, the ten thousandths digit 9. Repeat this process until the sum is a one-digit number, as demonstrated below.

$$\begin{array}{cccccccccc} 4 & . & 8 & \cancel{3}^{+1} & \cancel{9}^{+1} & \cancel{9}^{+1} & \cancel{9}^{+1} & 9 & 2 \\ 4 & . & 8 & 4 & 0 & 0 & 0 & & \end{array}$$

The number 4.8399992, rounded to the nearest hundred thousandth, is 4.84000. Notice that the zeroes at the end of the number are used to indicate that the number is accurate to the hundred thousandths place. While it is true that $4.84000 = 4.84$, when you are asked to round to a specific decimal place, it is customary to clarify your answer using the appropriate number of zeroes, so that the final digit appears in the place to which you are asked to round.

10.14 Round 99.9999999999 to the nearest billionth.

Decimal place value names work just like regular place value names (from Chapter 1), if you add "ths" at the end: tens, hundreds, thousands, millions, billions, trillions, etc.

Unlike digits left of the decimal point, decimal digits are not separated by commas into groups of three to assist in the identification of place values. It may help to name the decimal places in order, beginning with tenths, while counting to identify the digit to which you are rounding: (1) tenths, (2) hundredths, (3) thousandths, (4) ten thousandths, (5) hundred thousandths, (6) millionths, (7) ten millionths, (8) hundred millionths, (9) billionths, and (10) ten billionths. In this number comprised entirely of 9s, you are asked to round to the ninth decimal place.

Consider the digit immediately following the billionths place, the ten billionths place: 9. Because $9 \geq 5$, you add 1 to the billionths place, but like Problem 10.13, this causes a chain reaction. Adding 1 to 9 produces the two-digit number 10, so you insert a 0 into the billionths place and carry the 1 to the hundred millionths place, one digit to the left. This, in turn, causes a two-digit sum, and you repeat the process.

Each time you add 1 to a place value—including the place values on the integer side of the decimal point—that place value becomes 0 and the 1 is passed to the preceding place value. Finally, when that 1 is added to the 9 in the tens place, its migration ends in the hundreds place. The number 99.9999999999, rounded to the nearest billionth, is 100.000000000.

Adding and Subtracting Decimals

Keep the decimal points lined up

10.15 A student is asked to calculate the sum 2.63 + 1.8 and responds with the inaccurate answer displayed below. Identify the student's error and then compute the sum correctly.

$$\begin{array}{r} 2.{}^{7}\cancel{6}3 \\ +\ \ 1.8 \\ \hline 2.\ 8\ 1 \end{array}$$

The student correctly adds the digits in the right column: 3 + 8 = 11. The ones digit (1) is placed below the horizontal line, and the tens digit (1) is added to the upper digit in the middle column: 6 + 1 = 7. The student then adds the digits in the middle column correctly: 7 + 1 = 8. Finally, the student adds the numbers in the left column: 2 + 0 = 2.

While all of the columns have been correctly summed, the student is unable to arrive at the correct answer because the problem was set up incorrectly. Arguably the most critical step when adding or subtracting decimals is aligning their place values. In the student's work, the hundredths place 3 of 2.63 is aligned with the tenths place 8 of 1.8.

The most straightforward way to ensure that place values are correctly aligned is to align the decimal points—they should appear one above the other, points on a vertical line that runs north and south along the written page, as demonstrated below.

$$\begin{array}{r} 2.63 \\ +\ 1.8 \\ \hline \end{array}$$

Many teachers advise students to "fill in" missing place values with zeroes as a visual aid. Using place holder zero digits does not affect the values of the decimals, and you may find it more visually appealing.

$$\begin{array}{r} 2.63 \\ +\ 1.80 \\ \hline \end{array}$$

Add the columns of digits. Note that the sum in the tenths column is a two-digit number. You may choose to "carry the one" using the notation presented in Chapter 1 (writing the 1 above a column of numbers) or by explicitly adding 1 to the upper digit in the column (like the student in this problem).

$$\begin{array}{r} {}^{3}\cancel{2}.63 \\ +\ 1.80 \\ \hline 4.43 \end{array}$$

10.16 Calculate the sum: 46.19 + 7.03.

Align the decimal points to ensure that all of the place values appear in corresponding columns.

$$\begin{array}{r} 46.19 \\ +\ \ 7.03 \\ \hline \end{array}$$

The sum of the digits in the hundredths column is 9 + 3 = 12. Place the ones value of 12 below the horizontal line and add the tens value to the upper number in the tenths column.

$$\begin{array}{r} 46.{}^{2}\cancel{1}9 \\ +\ \ 7.\ 03 \\ \hline .\ 22 \end{array}$$

The sum of the values in the ones column again requires you to carry a ones digit, this time adding it to the 4 in the tens column.

$$\begin{array}{r} {}^{5}\cancel{4}6.19 \\ +\ \ 7.03 \\ \hline 53.22 \end{array}$$

The tenths places of both numbers line up, as do the hundredths places, etc.

10.17 Calculate the sum: 20.14 + 7.859.

Write the addition problem vertically, aligning the decimal points so that corresponding place values appear in the same column.

$$\begin{array}{r} 20.14 \\ +\ 7.859 \\ \hline \end{array}$$

You could write 7.859 on top and 20.14 underneath, but either way, you should still stick a 0 after the 4.

The lower number contains three decimal digits, whereas the upper number has only two. Insert a zero place holder so that both numbers have three decimal digits.

$$\begin{array}{r} 20.140 \\ +\ 7.859 \\ \hline \end{array}$$

Add the digits in each column.

$$\begin{array}{r} 20.140 \\ +\ 7.859 \\ \hline 27.999 \end{array}$$

10.18 Calculate the sum and round it to the nearest tenth: 24.01097 + 5.3816.

Write the problem vertically and align the decimal points. Insert a zero place holder at the end of 5.3816 to increase its number of decimal digits to five, matching the number of decimal digits in 24.01097.

$$\begin{array}{r} 24.01097 \\ +\ \ 5.38160 \\ \hline \end{array}$$

Add the values in each column; note that the sum in the ten thousandths column is a two-digit number, so 1 should be added to the thousandths place in the upper number.

$$\begin{array}{r} 24.01^{1}\cancel{0}97 \\ +\ \ 5.38\ 160 \\ \hline 29.39\ 257 \end{array}$$

Thus, 24.01097 + 5.38160 = 29.39257. The problem directs you to round to the nearest tenth. Notice that the hundredths digit is 9. Because $9 \geq 5$, you should add one to the tenths place and truncate the decimal at that digit. The answer, rounded to the nearest tenth, is 29.4.

10.19 Calculate the difference: 3.692 – 2.27.

Subtracting decimals requires the same initial setup as adding decimals: Ensure that the decimal points—and thus the place values of the numbers—are properly aligned. In this problem, 2.27 should be placed below 3.692, because 2.27 is subtracted from 3.692.

$$\begin{array}{r} 3.692 \\ -\ 2.27 \\ \hline \end{array}$$

Like decimal addition, decimal subtraction also uses zero place holders to ensure both numbers contain the same number of digits after the decimal point. In this case, a zero should be inserted after the 7 in the lower number.

$$\begin{array}{r} 3.692 \\ -\ 2.270 \\ \hline \end{array}$$

Subtract the lower digit from the upper digit in each column and record the difference below the horizontal line.

$$\begin{array}{r} 3.692 \\ -\ 2.270 \\ \hline 1.422 \end{array}$$

10.20 Calculate the difference: 19.428 – 2.063.

Align the decimal points and the corresponding place values; zero place holders are not necessary because both numbers contain three decimal digits.

$$\begin{array}{r} 19.428 \\ -\ \ 2.063 \\ \hline \end{array}$$

Subtract the lower digit from the upper digit in each column, beginning with the thousandths column.

$$\begin{array}{r} 19.428 \\ -\ \ 2.063 \\ \hline 5 \end{array}$$

If you need to review borrowing, flip back to Problem 2.13 and start there.

In order to subtract the digits in the hundredths column, you must first "borrow" one from the tenths column. Reduce the digit 4 by 1 and place that 1 in front of the hundredths digit 2, transforming it into the two-digit number 12.

$$\begin{array}{r} 19.\overset{3}{\cancel{4}}\,{}^{1}2\,8 \\ -\ \ 2.\ 0\ 6\,3 \\ \hline 5 \end{array}$$

Subtract the digits in the hundredths column (12 – 6 = 6) and the tenths column (3 – 0 = 3), as well as the digits on the left side of the decimal point.

$$\begin{array}{r} 19.\overset{3}{\cancel{4}}\,{}^{1}28 \\ -\ \ 2.\ 0\ 63 \\ \hline 17.\ 3\ 65 \end{array}$$

Thus, 19.428 – 2.063 = 17.365.

10.21 Calculate the difference: 5.93 – 4.275.

Align the place values of the decimals and insert a zero place holder to increase the number of decimal digits in 5.93 to match the number of decimal digits in 4.275.

$$\begin{array}{r} 5.930 \\ -\ 4.275 \\ \hline \end{array}$$

Subtract the digits in the thousandths column, borrowing one from the hundredths column.

$$\begin{array}{r} 5.9\,\overset{2}{\cancel{3}}\,{}^{1}0 \\ -\ 4.2\ 7\ 5 \\ \hline 5 \end{array}$$

In order to subtract the digits in the hundredths column, you must once again borrow, this time from the tenths column.

$$\begin{array}{r} 5.{}^{8}\cancel{9}{}^{12}\cancel{3}{}^{1}0 \\ -\ 4.\ 2\ \ 7\ 5 \\ \hline 1.\ 6\ \ 5\ 5 \end{array}$$

Thus, 5.93 – 4.275 = 1.655.

10.22 Calculate the difference: 7.2 – 10.83.

The number being subtracted is larger than the number from which it is subtracted: 10.83 > 7.2. Therefore, the difference will be a negative value. Now that the sign of the difference is known, you may ignore the signs of the numbers. Place the larger number (10.83) above the smaller number (7.2), aligning the decimal points and the corresponding place values. Insert a zero place holder after 7.2 to ensure that 7.2 and 10.83 have the same number of decimal digits.

$$\begin{array}{r} 10.83 \\ -\ \ 7.20 \\ \hline \end{array}$$

Calculate the differences of the digits in each column. You may elect to borrow a digit from the tens column to subtract the digits in the ones column or you may simply subtract the integers: 10 – 7 = 3.

$$\begin{array}{r} 10.83 \\ -\ \ 7.20 \\ \hline 3.63 \end{array}$$

Remember that the difference must be negative, as determined at the outset of the problem: 7.2 – 10.83 = –3.63.

10.23 Calculate the difference: 803.9 – 8.039.

Align the decimal points and the corresponding place values. Use zero place holders to ensure an equal number of decimal digits.

$$\begin{array}{r} 803.900 \\ -\ \ \ \ 8.039 \\ \hline \end{array}$$

To subtract the digits in the thousandths column, you must borrow from the hundredths column. However, you cannot borrow from 0, so you must first borrow from the tenths column, changing the digit 9 to 8 and transforming the 0 in the hundredths column into 10.

$$\begin{array}{r} 803.{}^{8}\cancel{9}{}^{1}0\,0 \\ -\ \ \ \ 8.\ 0\ 39 \\ \hline \end{array}$$

Now you can borrow from the 10 in the hundredths column, changing it to 9, in order to transform the 0 in the thousandths column into a 10.

$$\begin{array}{r} 803.{}^{8}\cancel{9}\,{}^{9}\cancel{10}\,{}^{1}0 \\ -\quad 8.\;0\quad 3\;\;9 \\ \hline \end{array}$$

Subtract the columns of decimal digits.

$$\begin{array}{r} 803.{}^{8}\cancel{9}\,{}^{9}\cancel{10}\,{}^{1}0 \\ -\quad 8.\;0\quad 3\;\;9 \\ \hline .\;8\quad 6\;\;1 \end{array}$$

In other words, the slashes and little floating numbers you used when subtracting the decimals are left out to make room for new slashes and floating numbers.

Repeat the above process to subtract the digits in the ones column: Borrowing from the tens column requires you to first borrow from the hundreds column. For clarity, the borrowing undertaken in the previous steps is omitted.

$$\begin{array}{r} {}^{7}\cancel{8}\,{}^{9}\cancel{10}\,{}^{1}3.900 \\ -\qquad 8.039 \\ \hline 7\quad 9\;5.861 \end{array}$$

Thus, 803.9 – 8.039 = 795.861.

Multiplying Decimals

Regular multiplication with a little counting thrown in

10.24 Calculate the product: 2(1.3).

Write the product vertically. Typically, the number with more digits is placed above the other. In this case, 1.3 is placed above 2.

$$\begin{array}{r} 1.3 \\ \times\; 2 \\ \hline \end{array}$$

Multiply the lower digit (2) by the rightmost digit in the top row: $2 \cdot 3 = 6$.

$$\begin{array}{r} 1.3 \\ \times\; 2 \\ \hline 6 \end{array}$$

Now move one digit to the left in the top number (1), and multiply it by the lower number: $2 \cdot 1 = 2$.

$$\begin{array}{r} 1.3 \\ \times\; 2 \\ \hline 2\;6 \end{array}$$

The problem is not yet complete; you still need to place the decimal point in the product. Count the number of digits that appear behind the decimal point in the original product 2(1.3). In this case, only one digit appears behind a decimal point, the digit 3. Therefore, only one digit in the answer should appear behind the decimal point: 2.6.

This is just like Problem 2.20, with a decimal stuck in.

10.25 Calculate the product: 5(7.406).

Write the product vertically, placing the number with more digits (7.406) above the number with fewer digits (5).

$$\begin{array}{r} 7.406 \\ \times \quad 5 \\ \hline \end{array}$$

When you multiply the lower number by the rightmost digit in the upper number, the result is a two-digit product: $6 \cdot 5 = 30$. Write the digit 0 below the horizontal line and place the digit 3 above the next digit in the top number, as you move from right to left.

$$\begin{array}{r} 3 \\ 7.406 \\ \times \quad 5 \\ \hline 0 \end{array}$$

Multiply the bottom number by the hundredths digit in the top number and add 3, the digit resulting from the last step, to the product: $(5 \cdot 0) + 3 = 0 + 3 = 3$.

$$\begin{array}{r} 3 \\ 7.406 \\ \times \quad 5 \\ \hline 30 \end{array}$$

Continue moving from right to left, multiplying each digit of the upper number by 5.

$$\begin{array}{r} 2\ \ 3 \\ 7.406 \\ \times \quad 5 \\ \hline 37\,030 \end{array}$$

There are a total of three digits behind the decimal point in the original product 5(7.406), so three digits of the answer calculated above should appear to the right of the decimal point.

$$5(7.406) = 37.030$$

You do not need to include zeroes that appear at the end of the decimal portion of a number. Therefore, 5(7.406) = 37.03.

10.26 Calculate the product: 0.6(0.9).

Write the product vertically. Both decimals contain one digit of significance (a single zero placed before a decimal point does not affect the value of a decimal any more than a single zero placed at the end of the decimal portion of the number).

$$\begin{array}{r} .6 \\ \times\ .9 \\ \hline \end{array}$$

Calculate the product of 6 and 9.

$$\begin{array}{r} .6 \\ \times\ .9 \\ \hline 54 \end{array}$$

A total of two digits appear after the decimal point in the original product 0.6(0.9)—one digit appears after the decimal point in the number 0.6 and one digit appears after the decimal point in the number 0.9. Therefore, two digits must appear after the decimal point in the final answer.

$$0.6(0.9) = 0.54$$

The "x" symbol is used for multiplication instead of the "·" symbol, because the "·" multiplication symbol can get confused with the decimal point.

10.27 Calculate the product: 4.3(1.6).

Begin by writing the product vertically.

$$\begin{array}{r} 4.3 \\ \times\ 1.6 \\ \hline \end{array}$$

Multiply the rightmost digit 6 of the lower number 1.6 by both digits of the upper number 4.3, moving from right to left. Carry digits when the results are two-digit numbers, as demonstrated in Problem 10.25. Review Problems 2.20–2.25 for additional help with the multiplication algorithm.

$$\begin{array}{r} \overset{1}{4}.3 \\ \times\ 1.6 \\ \hline 2\,5\,8 \end{array}$$

Insert a zero place holder before you move to the ones digit of the lower number.

$$\begin{array}{r} 4.3 \\ \times\ 1.6 \\ \hline 2\,5\,8 \\ 0 \end{array}$$

Now multiply each of the digits in the upper number by the ones digit of the lower number.

$$\begin{array}{r} 4.3 \\ \times\ 1.6 \\ \hline 258 \\ 430 \\ \hline \end{array}$$

Add the results between the horizontal lines.

$$\begin{array}{r} 4.3 \\ \times\ \ 1.6 \\ \hline 258 \\ +430 \\ \hline 688 \end{array}$$

A total of two digits appear behind the decimal points in the original product 4.3(1.6), so two digits of the final answer should appear behind the decimal point.

$$4.3(1.6) = 6.88$$

10.28 Calculate the product: 8.5(0.924).

Express the product vertically. The number with the most digits should be written above the other number.

$$\begin{array}{r} .924 \\ \times\ 8.5 \\ \hline \end{array}$$

Multiply the rightmost digit of the lower number by each digit of the upper number, moving from right to left and carrying digits when necessary.

$$\begin{array}{r} {\scriptstyle 1\,2\ } \\ .924 \\ \times\ 8.5 \\ \hline 4620 \end{array}$$

Repeat the process, this time using the ones digit in the lower number. Remember to include a zero place holder directly below the rightmost digit of the previous result, 4620.

$$\begin{array}{r} {\scriptstyle 1\,3\ \ } \\ .924 \\ \times\qquad 8.5 \\ \hline 4620 \\ 73920 \\ \hline \end{array}$$

Add the numbers between the horizontal lines.

$$\begin{array}{r} .924 \\ \times \quad 8.5 \\ \hline 4620 \\ 73920 \\ \hline 78540 \end{array}$$

Only drop that zero after you place the decimal point four digits in from the right.

A total of four digits appear after the decimal points in the original product 8.5(0.924), so four digits must appear after the decimal point in the final answer.

$$8.5(0.924) = 7.8540$$

The final zero at the end of the number should be omitted: 8.5(0.924) = 7.854.

10.29 Calculate the product: 3.1(0.0007).

Write the product vertically.

$$\begin{array}{r} .0007 \\ \times \quad 3.1 \\ \hline \end{array}$$

Apply the process explained in Problems 10.27–10.28, multiplying each of the digits of the upper number by each digit of the lower number in an orderly fashion.

$$\begin{array}{r} .0007 \\ \times \quad 3.1 \\ \hline 0007 \\ +\ 00210 \\ \hline 217 \end{array}$$

A total of five digits appear behind the decimal points in the original product 3.1(0.0007), so five digits must appear behind the decimal point in the final answer. However, the answer contains only three digits. In order to accommodate the required number of decimal places behind the decimal point, you must add two zeroes at the beginning of the number, changing 217 to 00217.

$$3.1(0.0007) = 0.00217$$

10.30 Calculate the product: 2.6(40.132).

Express the product vertically.

$$\begin{array}{r} 40.132 \\ \times \quad 2.6 \\ \hline \end{array}$$

Multiply the right digit of the lower number by each digit of the upper number, working from right to left.

$$\begin{array}{r} {}^{11} \\ 40.132 \\ \times \quad 2.6 \\ \hline 240792 \end{array}$$

Repeat the process for the remaining digit in the lower number; remember to include the zero placeholder.

$$\begin{array}{r} 40.132 \\ \times \quad 2.6 \\ \hline 240792 \\ +802640 \\ \hline \end{array}$$

This is the zero placeholder the book is talking about.

Calculate the sum of the numbers between the horizontal lines.

$$\begin{array}{r} 40.132 \\ \times \quad 2.6 \\ \hline 240792 \\ +802640 \\ \hline 1043432 \end{array}$$

A total of four digits appear right of the decimal points in the original product 2.6(40.132), so four digits should appear right of the decimal point in the final answer as well.

$$2.6(40.132) = 104.3432$$

10.31 Calculate the product: 7.58(19.3).

Compute the product, applying the procedure described in the preceding problems.

$$\begin{array}{r} 7.58 \\ \times \ 19.3 \\ \hline 2274 \\ 68220 \\ +75800 \\ \hline 146.294 \end{array}$$

There are three digits in the lower number (19.3), so there are three sets of digits between the horizontal lines. There's one zero placeholder in the second line and two zero placeholders in the third line.

10.32 Simplify the expression: 4.5(1.2) + 8.9607.

According to the order of operations, multiplication should be completed before addition. Begin by calculating 4.5(1.2).

$$\begin{array}{r} 4.5 \\ \times \ 1.2 \\ \hline 90 \\ +450 \\ \hline 5.40 \end{array}$$

Replace the product in the original expression with its actual value.

$$4.5(1.2) + 8.9607 = 5.40 + 8.9607$$

Now calculate the sum, aligning the decimal points and the corresponding place values.

$$\begin{array}{r} 5.4000 \\ +\ 8.9607 \\ \hline 14.3607 \end{array}$$

Thus, $4.5(1.2) + 8.9607 = 14.3607$.

Look at Problems 2.29–2.35 if you need to review, but the book does hold your hand a little bit during this problem.

Dividing Decimals

Long division is back—with a vengeance!

10.33 Calculate the quotient: $8.46 \div 2$.

In order to divide decimals, you must first understand the process of long division. The solutions in this section assume that you have first mastered that procedure. Begin by writing the divisor in front of, and the dividend within, a long division symbol.

$$2\overline{)8.46}$$

Before you begin dividing, place a decimal point above the horizontal line so that it aligns with the decimal point in the dividend.

$$\begin{array}{r} . \\ 2\overline{)8.46} \end{array}$$

Divide the first digit of the dividend (8) by the divisor (2). Record the result ($8 \div 4 = 2$) above the dividend digit.

$$\begin{array}{r} 4. \\ 2\overline{)8.46} \end{array}$$

Because $8 \div 2 = 4$, it follows that $4 \cdot 2 = 8$; the number above the horizontal line multiplied by the divisor is equal to 8. Record this value beneath the dividend and subtract.

$$\begin{array}{r} 4. \\ 2\overline{)8.46} \\ -8 \\ \hline 0 \end{array}$$

Copy the next digit of the dividend (4) next to the difference you just calculated.

$$\begin{array}{r} 4. \\ 2\overline{)\,8.46} \\ -8 \\ \hline 0\,4 \end{array}$$

Repeat the process, dividing the number below the horizontal line (4) by the divisor (2), recording the result above the horizontal line ($4 \div 2 = 2$), multiplying that value by the divisor ($2 \cdot 2 = 4$), writing the product below the digit 4, and subtracting.

$$\begin{array}{r} 4.2 \\ 2\overline{)\,8.46} \\ -8 \\ \hline 0\,4 \\ -\,4 \\ \hline 0 \end{array}$$

Complete the division problem by copying the last digit of the dividend (6) below the difference you just calculated and repeat the process.

$$\begin{array}{r} 4.23 \\ 2\overline{)\,8.46} \\ -8 \\ \hline 0\,4 \\ -\,4 \\ \hline 06 \\ -\,6 \\ \hline 0 \end{array}$$

NOW the long division review is over, so if you don't understand how long division works, make sure you go back and review those problems in Chapter 2.

You conclude that $8.46 \div 2 = 4.23$.

10.34 Calculate the quotient: $0.92 \div 4$.

Use long division notation to write the quotient and copy the decimal point above the horizontal line. Note that 4 divides into 0, the first digit of the dividend, 0 times. In fact, any non-zero integer divides into 0 exactly 0 times.

$$\begin{array}{r} 0. \\ 4\overline{)0.92} \end{array}$$

Divide the second digit of the dividend by the divisor: $9 \div 4 = 2 \text{ r } 1$. Place the quotient (2) above the horizontal line. When you multiply that 2 by the divisor and subtract, the result is the remainder (1).

$$\begin{array}{r} 0.2 \\ 4\overline{)\,0.92} \\ -8 \\ \hline 1 \end{array}$$

Copy the remaining digit of the dividend next to the difference you just calculated and complete the problem.

$$\begin{array}{r} 0.23 \\ 4\overline{)\,0.92} \\ -8 \\ \hline 12 \\ -12 \\ \hline 0 \end{array}$$

You conclude that $0.92 \div 4 = 0.23$.

10.35 Calculate the quotient: $47.91 \div 3$.

Apply the process described in Problems 10.33–10.34.

$$\begin{array}{r} 15.97 \\ 3\overline{)\,47.91} \\ -3 \\ \hline 17 \\ -15 \\ \hline 29 \\ -27 \\ \hline 21 \\ -21 \\ \hline 0 \end{array}$$

You conclude that $47.91 \div 3 = 15.97$.

10.36 Calculate the quotient: $0.2 \div 5$.

Rewrite the quotient using long division notation. Copy the decimal point above the horizontal line and note that 5 divides into 0 exactly 0 times.

$$\begin{array}{r} 0. \\ 5\overline{)0.2} \end{array}$$

You are now faced with a dilemma: The next digit in the dividend (2) is less than the divisor (5), so you must record a zero above divisor digit 2.

$$\begin{array}{r} 0.0 \\ 5\overline{)0.2} \end{array}$$

If you were dividing integers, this would signify the end of the problem—a final dividend digit less than the divisor represents the remainder. However, this is *not* the case in decimal division, where remainders are not traditionally applied. Instead, you place a zero at the end of the dividend and continue to long divide. (Recall that zeroes placed after the decimal point at the end of a number do not affect the number's value.)

Remember this! Decimal long division = no remainders.

$$5\overset{\displaystyle 0.0}{\overline{)0.20}}$$

The dilemma is now resolved; 20 is evenly divisible by 5.

$$\begin{array}{r} 0.04 \\ 5\overline{)0.20} \\ \underline{-20} \\ 0 \end{array}$$

Now that you have recorded a difference of zero and no digits in the dividend remain, the long division problem is concluded: $0.2 \div 5 = 0.04$. The quotient 0.04 is described as a "terminating decimal," because it—like the preceding quotients in this section—contains a finite number of digits.

Finite is the opposite of infinite. There aren't an infinite number of digits in 0.04 (it doesn't go on and on and on forever), so there are a finite number of digits.

10.37 Calculate the quotient and verify your answer: $6.5 \div 1.3$.

When a divisor contains decimal digits, the long division process includes an additional step. Move the decimal point to the right, counting the number of digits as you move it, until all of its digits are left of the decimal point. In this case, you move the decimal point one place to the right, transforming 1.3 into 13.0, or simply 13. You must also move the decimal point within the dividend the same number of places. Thus, 6.5 becomes 65.0, or 65.

$$13\overline{)65.}$$

The newly placed decimal point is typically omitted from the divisor, but you may choose to include it at the end of the dividend. Like Problem 10.36, the solution may require you to add zeroes to the end of the dividend, so including the decimal in the initial step ensures that it will be located in the correct place later if needed.

The dividend is evenly divisible by the divisor: $65 \div 13 = 5$.

$$\begin{array}{r} 5. \\ 13\overline{)\ 65.} \\ \underline{-65} \\ 0 \end{array}$$

Therefore, $6.5 \div 1.3 = 5$. Verify the solution by multiplying the divisor and quotient. If your answer is correct, the result will be the dividend.

$$\begin{array}{r} 1.3 \\ \underline{\times\ 5} \\ 6.5 \end{array}$$

10.38 Calculate the quotient: $4.266 \div 1.8$.

Rewrite the quotient using long division notation.

$$1.8\overline{)4.266}$$

Move the decimal point one place to the right in the divisor to eliminate the decimal digits; move the decimal point in the dividend one place to the right as well.

18(2) = 36
and 42 − 36 = 6, so
42 ÷ 18 = 2 r 6.

$$\begin{array}{r} . \\ 18\overline{)42.66} \end{array}$$

Begin the long division process by calculating the number of whole times 18 divides into 42.

$$\begin{array}{r} 2. \\ 18\overline{)\ 42.66} \\ \underline{-36} \\ 6 \end{array}$$

Copy the next digit of the dividend (6) next to the difference you just calculated. Continue the long division process.

$$\begin{array}{r} 2.3 \\ 18\overline{)\ 42.66} \\ \underline{-36} \\ 66 \\ \underline{-54} \\ 12 \end{array}$$

Copy the final digit of the dividend (6) next to the most recent difference (12) and repeat the process.

$$\begin{array}{r} 2.37 \\ 18\overline{)\ 42.66} \\ \underline{-36} \\ 66 \\ \underline{-54} \\ 126 \\ \underline{-126} \\ 0 \end{array}$$

You conclude that $4.266 \div 1.8 = 2.37$.

10.39 Calculate the quotient: $19.837 \div 2.39$.

Write the quotient using long division notation. The decimal point in the divisor should be moved two places to the right, so that the result is a whole number with no decimal digits. As a result, the decimal point in the dividend should be moved two places to the right as well.

$$239\overline{)1983.7}$$

The first three digits of the dividend (198) comprise a number that is less than the divisor (239), so you should divide 239 into the first four digits of the dividend.

$$\begin{array}{r} 8. \\ 239\overline{)\ 1983.7} \\ \underline{-1912} \\ 71 \end{array}$$

Multiply 239 by larger and larger integers (2, 3, 4, 5, etc.) until you get the biggest possible number that's still less than 1,983.

239(7) = 1,673
239(8) = 1,912
239(9) = 2,151

Looks like you should go with 8, because 9 is too large: 2,151 > 1,983.

Copy the next digit of the dividend (7) next to the difference (71) and repeat the process.

$$\begin{array}{r} 8.3 \\ 239\overline{)\ 1983.7} \\ \underline{-1912} \\ 71\,7 \\ \underline{-71\,7} \\ 0 \end{array}$$

All of the digits in the dividend have been addressed and the final difference is zero. Therefore, the quotient is a terminating decimal: 19.837 ÷ 2.39 = 8.3.

10.40 Calculate the quotient: 9.6 ÷ 1.8.

Express the quotient using long division notation, moving the decimal points in the divisor and the dividend one place to the right.

$$18\overline{)96.}$$

Begin the long division process, identifying the largest number by which you can multiply 18 to obtain a product less than or equal to 96.

$$\begin{array}{r} 5 \\ 18\overline{)\ 96.} \\ \underline{-90} \\ 6 \end{array}$$

The difference does not equal zero, so you must place an additional zero at the end of the dividend. Copy that 0 next to the difference you calculated and continue dividing.

$$\begin{array}{r} 5.3 \\ 18\overline{)\ 96.0} \\ \underline{-90} \\ 6\,0 \\ \underline{-\ 5\,4} \\ 6 \end{array}$$

Once again, the remainder is a non-zero number. Place *another* zero at the end of the dividend and continue dividing.

$$\begin{array}{r} 5.33 \\ 18\overline{)\ 96.00} \\ \underline{-90} \\ 6\,0 \\ \underline{-5\,4} \\ 60 \\ \underline{-\ 54} \\ 6 \end{array}$$

You're right back where you started. You started with a 6 and ended with a 6, and every time you drop down another 0 from the dividend, you'll end up with another 6.

Each time a 0 is placed at the end of the divisor, a difference of 6 is transformed into the number 60. You then multiply 18 by 3 to get 54, subtract it from 60, and get a result of 6. You could repeat this process infinitely, and each time a zero is placed at the end of the dividend, the result is a difference of 6.

You conclude that $9.6 \div 1.8 = 5.333333333....$ This is described as a "repeating decimal," because the same decimal digit repeats endlessly; continuing to add zeroes to the dividend in hopes of identifying a terminating decimal is a pointless exercise, because each decimal place will be equal to 3.

Repeating decimals are usually written using shorthand notation; a small horizontal bar is placed above the digit or digits that repeat infinitely: $5.\overline{3} = 5.333333333...$.

10.41 Calculate the quotient and express it as a decimal: $5 \div 11$.

Rewrite the expression using long division notation. Like Problem 10.36, this divisor is greater than the dividend. Therefore, you should place an additional zero at the end of the dividend. Ensure that the decimal point is also included, as the value of the dividend cannot change. It is appropriate to rewrite 5 as 5.0, but omitting the decimal produces unequal values: $5 \neq 50$.

$$\begin{array}{r} . \\ 11\overline{)5.0} \end{array}$$

Treat 5.0 as though it were the number 50; now that the decimal point is correctly placed above the horizontal line, pay no attention to it in the dividend. The divisor (11) divides into 50 a total of 4 times with a remainder of 6.

$$\begin{array}{r} .4 \\ 11\overline{)\ 5.0} \\ \underline{-4\,4} \\ 6 \end{array}$$

The difference is not equal to zero, so place an additional zero at the end of the dividend and continue the long division process.

$$
\begin{array}{r}
.45 \\
11\overline{)\ 5.00} \\
\underline{-4\,4} \\
60 \\
\underline{-\,55} \\
5
\end{array}
$$

Continue to place additional zeroes at the end of the dividend until the decimal either repeats (like Problem 10.40) or terminates (like Problem 10.38).

$$
\begin{array}{r}
.4545 \\
11\overline{)\ 5.0000} \\
\underline{-4\,4} \\
60 \\
\underline{-55} \\
50 \\
\underline{-\,44} \\
60 \\
\underline{-55} \\
5
\end{array}
$$

The digits of the quotient will repeat infinitely: 0.4545454545.... Because the quotient is a repeating decimal, you can conclude the long division process: $5 \div 11 = 0.\overline{45}$. Note that a horizontal bar is placed over both of the digits 4 and 5 in the answer, because both digits repeat.

10.42 Calculate the quotient: $4.3 \div 1.6$.

Write the quotient as a long division problem, moving the decimal points one place to the right in both the divisor and the dividend. Place zeroes at the end of the dividend until the quotient either repeats infinitely or terminates.

$$
\begin{array}{r}
2.6875 \\
16\overline{)\ 43.0000} \\
\underline{-32} \\
11\,0 \\
\underline{-9\,6} \\
1\,40 \\
\underline{-1\,28} \\
120 \\
\underline{-\ 112} \\
80 \\
\underline{-\,80} \\
0
\end{array}
$$

The quotient terminates at the ten thousandths place, and you conclude that $4.3 \div 1.6 = 2.6875$.

Chapter 11

CONVERTING BETWEEN FRACTIONS AND DECIMALS

Here comes the long division

The four preceding chapters have investigated fractions and decimals, including the application of the four primary binary operations to both. In this chapter, you will learn how to convert between fractions and decimals, including a method that converts a repeating decimal into a fraction regardless of the number of decimal digits that repeat. Make sure you complete the sections of Chapters 2 and 10 that address long division, as that is a prerequisite skill for the vast majority of the problems in this chapter.

Do you like long division? No matter what you actually answered, I am going to pretend that you said, "Of course I do!" Good, because this chapter is chock full of long division. Are those tears of joy I see? No? Just the regular kind of tears? Understandable.

I do have a piece of good news. Even though there are lots of long division problems ahead, the processes you'll learn in this chapter are very repetitive. Very, VERY repetitive. There's a lot of practice but not a lot of new concepts until Chapter 12, so at least there's that.

Expressing Fractions as Decimals

Divide the top number by the bottom number

11.1 Express the fraction as a decimal: $\frac{1}{2}$.

To convert a fraction into a decimal, express the fraction as a quotient and then apply decimal long division. In this problem, $\frac{1}{2} = 1 \div 2$; express the quotient using long division notation.

$$2\overline{)1}$$

The divisor (2) is greater than the dividend (1). As first demonstrated in Problem 10.41, you should place a decimal point and a zero at the end of the divisor.

This is allowed because 1.0 and 1 are equal values. You can stick a decimal point after an integer and then write as many zeroes as you want after the decimal point. It won't affect the integer's value.

$$\begin{array}{r} . \\ 2\overline{)1.0} \end{array}$$

Notice that 2 divides into 10 a total of 5 times. If you are unsure how to divide decimal values, review Problems 10.33–10.42.

$$\begin{array}{r} .5 \\ 2\overline{)\ 1.0} \\ \underline{-1\,0} \\ 0 \end{array}$$

You can write the answer as .5 or 0.5. Either one is correct. Math books don't like the empty space that's left of the decimal point, so they stick a zero in there.

The quotient is .5, which is commonly written 0.5 in textbooks—given a decimal with no whole place values, a zero place holder is inserted left of the decimal point. You conclude that $\frac{1}{2} = 0.5$.

11.2 Express the fraction as a decimal: $\frac{3}{4}$.

Apply long division to calculate the quotient of the numerator and denominator. Because the divisor is greater than the dividend, you will need to place a decimal point and a zero at the end of the dividend.

$$\begin{array}{r} . \\ 4\overline{)3.0} \end{array}$$

Temporarily ignore the decimal point in the dividend, treating 3.0 as the integer 30 while you divide.

$$\begin{array}{r} .7 \\ 4\overline{)\ 3.0} \\ \underline{-2\,8} \\ 2 \end{array}$$

The problem is not complete until the decimal in the quotient either repeats infinitely or terminates. Place another zero at the end of the divisor and continue the long division process.

$$\begin{array}{r} .75 \\ 4\overline{)\;3.00} \\ \underline{-2\,8} \\ 20 \\ \underline{-20} \\ 0 \end{array}$$

If you did, you'd be subtracting 0 from 0, getting 0, and repeating forever and ever. In other words, you'd get 0.75000000000... .

Because the difference is equal to zero, the decimal terminates; there is no reason to continue to place zeroes at the end of the dividend. You conclude that $\frac{3}{4} = 0.75$.

11.3 Express the fraction as a decimal: $\frac{2}{3}$.

Apply the procedure described in Problems 11.1–11.2: Express the fraction as a quotient and use long division to calculate that quotient.

$$\begin{array}{r} .6 \\ 3\overline{)\;2.0} \\ \underline{-1\,8} \\ 2 \end{array}$$

Like Problem 11.2, the first difference you calculate is not equal to zero. Therefore, you should place additional zeroes at the end of the dividend until it is clear that the decimal quotient either repeats or terminates.

$$\begin{array}{r} .666 \\ 3\overline{)\;2.000} \\ \underline{-1\,8} \\ 20 \\ \underline{-18} \\ 20 \\ \underline{-18} \\ 2 \end{array}$$

Each time a zero is placed after the decimal point in the dividend, it results in a place value of 6 in the quotient. Therefore, this is a repeating decimal and you conclude that $\frac{2}{3} = 0.666666...$ or, more succinctly, $\frac{2}{3} = 0.\overline{6}$.

Any digit or digits under this bar repeat infinitely. That means the decimal $0.\overline{321}$ is equal to 0.321321321321... .

11.4 Express the fraction as a decimal: $\frac{5}{8}$.

Apply long division to calculate the quotient of the numerator and denominator.

$$\begin{array}{r} .6 \\ 8\overline{)\;5.0} \\ -4\,8 \\ \hline 2 \end{array}$$

Continue placing zeroes at the end of the dividend until the quotient either repeats or terminates.

$$\begin{array}{r} .625 \\ 8\overline{)\;5.000} \\ -4\,8 \\ \hline 20 \\ -16 \\ \hline 40 \\ -40 \\ \hline 0 \end{array}$$

Because you calculate a final difference of zero—rather than falling into a repeating pattern that leads to an infinitely repetitive string of digits, like in Problem 11.3—you conclude that $\frac{5}{8} = 0.625$.

11.5 Express the fraction as a decimal: $\frac{7}{9}$.

Divide the numerator by the denominator until either a terminating or repeating decimal emerges as the quotient.

$$\begin{array}{r} .777 \\ 9\overline{)\;7.000} \\ -6\,3 \\ \hline 70 \\ -63 \\ \hline 70 \\ -63 \\ \hline 7 \end{array}$$

Each time you place a zero at the end of the dividend, it results in the same difference, which leads to a place value of 7 in the quotient. Therefore, the result is a repeating decimal and you conclude that $\frac{7}{9} = 0.\overline{7}$.

Any number over 9 is equal to that number repeating itself forever as a decimal:

$\frac{1}{9} = 0.\overline{1}$

$\frac{2}{9} = 0.\overline{2}$

$\frac{3}{9} = 0.\overline{3}$

$\frac{4}{9} = 0.\overline{4}$

etc.

11.6 Express the fraction as a decimal: $\frac{9}{5}$.

Unlike Problems 11.1–11.5, you are given an improper fraction. This does not change the procedure used to convert the fraction into a decimal, but you may notice one appreciable difference in the result. Because the divisor is no longer greater than the dividend, the quotient will be greater than one—at least one non-zero digit will appear left of the decimal point.

$$\begin{array}{r}
1 \\
5\overline{)\ 9} \\
\underline{-5} \\
4
\end{array}$$

You are instructed to report the answer as a decimal, so you cannot stop and conclude that $9 \div 5 = 1$ r 4. Instead, place a decimal point and a zero at the end of the dividend and continue the long division process until the decimal either repeats or terminates.

$$\begin{array}{r}
1.8 \\
5\overline{)\ 9.0} \\
\underline{-5} \\
4\ 0 \\
\underline{-4\ 0} \\
0
\end{array}$$

The quotient is a terminating decimal; you conclude that $\frac{9}{5} = 1.8$.

11.7 Express the fraction as a decimal: $\frac{23}{4}$.

Like Problem 11.6, this problem requires you to place a decimal point and additional zeroes at the right side of the dividend in order to report the quotient as a decimal value.

$$\begin{array}{r}
5.75 \\
4\overline{)\ 23.00} \\
\underline{-20} \\
3\ 0 \\
\underline{-2\ 8} \\
20 \\
\underline{-20} \\
0
\end{array}$$

You conclude that $\frac{23}{4} = 5.75$.

11.8 Verify your answer to Problem 11.7 by expressing $\frac{23}{4}$ as a mixed number and then converting the proper fraction portion of the mixed number into a decimal.

Divide 23 by 4. The quotient is the whole number, the remainder is the proper fraction numerator, and the proper fraction denominator is the same as the improper fraction denominator.

Rewrite the improper fraction as a mixed number.

$$\frac{23}{4} = 5\frac{3}{4}$$

Recall that a mixed number is equal to the sum of its whole and fractional parts.

$$5\frac{3}{4} = 5 + \frac{3}{4}$$

Replace the fraction with its decimal equivalent. According to Problem 11.2, $\frac{3}{4} = 0.75$.

$$\begin{aligned} 5\frac{3}{4} &= 5 + 0.75 \\ &= 5.75 \end{aligned}$$

The result is exactly equal to the decimal representation of $\frac{23}{4}$ presented in Problem 11.7, thus verifying that the solution is 5.75.

11.9 Express the fraction as a decimal: $\frac{4}{11}$.

Apply long division to calculate the quotient $4 \div 11$, and continue until that decimal quotient either repeats infinitely or terminates.

$$\begin{array}{r} .3636 \\ 11\overline{)\ 4.0000} \\ \underline{-3\ 3} \\ 70 \\ \underline{-66} \\ 40 \\ \underline{-33} \\ 70 \\ \underline{-66} \\ 4 \end{array}$$

Consider the sequence of consecutive differences in the long division problem: 7, 4, 7, 4. This pattern will continuously repeat as you place zeroes at the end of the dividend. Therefore, the corresponding digits in the quotient (3, 6, 3, 6) will repeat as well. You conclude that the quotient is a repeating decimal.

$$\frac{4}{11} = 0.\overline{36}$$

Note that the horizontal bar specifying the repeating decimal digits extends over 3 and 6, as the entire string "36" repeats infinitely.

11.10 Express the fraction as a decimal: $\frac{14}{15}$.

Compute the quotient $14 \div 15$ using long division.

$$\begin{array}{r} .9333 \\ 15\overline{)\ 14.0000} \\ \underline{-13\ 5} \\ 50 \\ \underline{-45} \\ 50 \\ \underline{-45} \\ 50 \\ \underline{-45} \\ 5 \end{array}$$

Not all of the digits have to repeat. They just have to settle into some sort of repeating pattern eventually.

Consider the sequence of differences calculated above: 5, 5, 5, 5. Each time you place a zero at the end of the dividend, it produces a difference of 5. Therefore, the quotient is a repeating decimal. You conclude that $\frac{14}{15} = 0.9\overline{3}$. Notice that the horizontal bar is placed only above the digit 3. Although the tenths digit 9 is part of the quotient, it does not repeat infinitely.

11.11 Express the fraction as a decimal: $\frac{56}{125}$.

Compute the quotient $56 \div 125$ by applying long division.

$$\begin{array}{r} .44 \\ 125\overline{)\ 56.00} \\ \underline{-50\ 0} \\ 6\ 00 \\ \underline{-5\ 00} \\ 1\ 00 \end{array}$$

You may be tempted to conclude that the quotient is equal to $0.\overline{4}$, as you have identified two consecutive decimal place values that are equal. Notice, however, that the sequence of differences in the long division problem do not repeat infinitely: 60, 100; for the moment, you cannot conclude that this quotient is a repeating decimal. Place an additional zero at the end of the dividend and continue dividing.

$$
\begin{array}{r}
.448 \\
125\overline{)\;56.000} \\
\underline{-50\,0} \\
6\,00 \\
\underline{-5\,00} \\
1\,000 \\
\underline{-\;1\,000} \\
0
\end{array}
$$

Problem 11.6 converts the reciprocal of $\frac{5}{9}$ into a decimal. The results are totally different. That's one way to prove that division is not commutative. In other words, $5 \div 9 \neq 9 \div 5$.

The result is a terminating decimal: $\frac{56}{125} = 0.448$.

11.12 Express the mixed number as a decimal: $10\frac{5}{9}$.

The whole number portion of the mixed number will also serve as the whole number portion of the decimal, so 10 will appear left of the decimal point. Compute the quotient $5 \div 9$ to identify the digits right of the decimal point.

$$
\begin{array}{r}
.555 \\
9\overline{)\;5.000} \\
\underline{-4\,5} \\
50 \\
\underline{-45} \\
50 \\
\underline{-45} \\
5
\end{array}
$$

The sequence of differences in the long division problem is 5, 5, 5; adding additional zeroes to the dividend will result in a continued string of 5s. Thus, you conclude that the quotient is a repeating decimal. Remember to include the whole number portion 10 of the decimal: $10\frac{5}{9} = 10.\overline{5}$.

11.13 Express the fraction as a decimal: $\frac{16}{37}$.

Apply long division to determine whether the quotient $16 \div 37$ is a repeating or terminating decimal.

$$
\begin{array}{r}
.4324 \\
37\overline{)\;16.0000} \\
\underline{-14\,8} \\
1\,20 \\
\underline{-1\,11} \\
90 \\
\underline{-74} \\
160 \\
\underline{-\;148} \\
12
\end{array}
$$

One of the digits in the quotient (4) has repeated. It is possible that the 4 continues to repeat infinitely, or perhaps the entire string of digits 432 will repeat. Continue dividing to determine which occurs.

$$\begin{array}{r}
.432432 \\
37\overline{)\ 16.000000} \\
\underline{-14\,8} \\
1\,20 \\
\underline{-1\,11} \\
90 \\
\underline{-74} \\
160 \\
\underline{-148} \\
120 \\
\underline{-111} \\
90 \\
\underline{-74} \\
16
\end{array}$$

The sequence of differences repeats: 12, 9, 16, 12, 9, 16. Therefore, the quotient is a repeating decimal: $\frac{16}{37} = 0.\overline{432}$.

11.14 Express the fraction as a decimal: $\frac{17}{222}$.

Apply long division to identify the decimal representation of $17 \div 222$. Note that you will have to place a decimal point and *two* zeroes behind the dividend; placing only one zero results in the quotient $170 \div 222$, in which case the divisor is still greater than the dividend. Thus, the first digit of the quotient should be recorded above the second zero in the dividend, in the hundredths place.

$$\begin{array}{r}
.0765 \\
222\overline{)\ 1\,7.0000} \\
\underline{-1{,}5\,54} \\
1460 \\
\underline{-1332} \\
1280 \\
\underline{-1110} \\
170
\end{array}$$

Although none of the digits in the quotient have repeated, the last difference you calculate is equal to 170. Placing a zero at the end of the dividend and copying that zero next to the difference—the next step in the long division process—results in 1,700. Not coincidentally, these are the first four digits of the dividend, and also constitute the first number into which you divided 222.

Thus, you can conclude that the decimal digits of the quotient will repeat. If you are not convinced, continue dividing to confirm that the digits 765 will repeat infinitely, as demonstrated below.

$$\begin{array}{r} .0765765 \\ 222\overline{)\ 17.0000000} \\ \underline{-15\,54} \\ 1\,460 \\ \underline{-1\,332} \\ 1280 \\ \underline{-1110} \\ 1700 \\ \underline{-1554} \\ 1460 \\ \underline{-1332} \\ 1280 \\ \underline{-1110} \\ 170 \end{array}$$

So don't put zero underneath the "these digits repeat" horizontal bar.

The tenths digit of the quotient (0) does not repeat; the hundredths, thousandths, and ten thousandths digits constitute the repeating sequence. You conclude that $\frac{17}{222} = 0.0\overline{765}$.

11.15 Express the fraction as a decimal: $\frac{8}{13}$.

Apply long division until the decimal in the quotient either terminates or repeats.

Don't give up. This problem takes a while—it won't repeat or terminate after two or three digits like the other problems in this section.

$$\begin{array}{r} .6153846 \\ 13\overline{)\ 8.0000000} \\ \underline{-7\,8} \\ 20 \\ \underline{-13} \\ 70 \\ \underline{-65} \\ 50 \\ \underline{-39} \\ 110 \\ \underline{-104} \\ 60 \\ \underline{-52} \\ 80 \\ \underline{-78} \\ 2 \end{array}$$

Consider the sequence of differences calculated above: 2, 7, 5, 11, 6, 8, 2. The final value is 2, which matches the first difference; the sequence will infinitely repeat. Therefore, you conclude $\frac{8}{13} = 0.\overline{615384}$.

Expressing Decimals as Fractions

Denominators are powers of 10—well, mostly

11.16 Express the decimal as a fraction: 0.7.

The fraction $\frac{1}{10}$ is read "one tenth," which is (not coincidentally) the same way to read the decimal 0.1. Furthermore, if you apply long division, you can verify that $\frac{1}{10} = 1 \div 10 = 0.1$.

$$\begin{array}{r} .1 \\ 10\overline{)\,1.0} \\ -1\,0 \\ \hline 0 \end{array}$$

Similarly, the fraction one hundredth $\left(\frac{1}{100}\right)$ is equal to the decimal with a value of 1 in the hundredths place: 0.01. These observations are the foundations upon which the decimal-to-fraction conversion process is based.

The decimal 0.7 contains exactly one digit right of the decimal point, the digit 7 in the tenths place. The equivalent fraction contains the digit 7 in the numerator and the value 10 in the denominator: $0.7 = \frac{7}{10}$.

11.17 Express the decimal as a fraction: 0.6.

One digit appears right of the decimal point, in the tenths place, so that digit (6) should appear in the numerator and 10 should appear in the denominator.

$$0.6 = \frac{6}{10}$$

To complete the problem, reduce the fraction to lowest terms.

$$\frac{6 \div 2}{10 \div 2} = \frac{3}{5}$$

You conclude that $0.6 = \frac{3}{5}$.

11.18 Express the decimal as a fraction: 0.29.

Two digits appear right of the decimal point; the rightmost digit appears in the hundredths place. Place both digits in the numerator of the fraction and 100 in the denominator.

$$0.29 = \frac{29}{100}$$

Powers of 10 are numbers that start with 1 and end with zeroes: 10; 100; 1,000; 10,000; 100,000; 1,000,000; etc. They are called powers of 10 because they are equal to 10 raised to different exponential powers: $10^2 = 100$, $10^3 = 1{,}000$, etc.

There are two techniques you can apply to choose the correct denominator when converting decimals to fractions:

- Choose the power of 10 that corresponds with the name of the rightmost digit. In this problem, the rightmost digit appears in the hundredths place, so the denominator is one hundred.
- The denominator should start with the digit 1 followed by as many zeroes as there are digits right of the decimal point. In this case, there are two digits right of the decimal point, so the denominator is 100, the digit 1 followed by two zeroes. When you are counting digits right of the decimal point, do not include zeroes appearing at the end of the number. For example, 0.29 and 0.2900000 both contain two digits after the decimal point, because the zeroes at the end of the latter decimal are unnecessary.

11.19 Express the decimal as a fraction: 0.050.

Although three digits appear after the decimal point, the final digit is a zero; it can be omitted without changing the value of the decimal: $0.050 = 0.05$. Note that the zero between the decimal point and the digit 5 cannot be eliminated, because doing so would affect the value of the fraction: $0.05 \neq 0.5$.

To express 0.05 as a fraction, note that two digits appear right of the decimal point, ending in the hundredths place. Those digits comprise the numerator of the equivalent fraction, and the denominator is 100, based upon the name of the rightmost decimal place.

You couldn't do this when the decimal point was still in there (0.05), but now that the decimal is gone, writing 5 as 05 doesn't make much sense.

$$0.050 = \frac{05}{100}$$

Omit the 0 preceding the digit 5 in the numerator and reduce the fraction to lowest terms.

$$\begin{aligned}\frac{05}{100} &= \frac{5}{100}\\ &= \frac{5 \div 5}{100 \div 5}\\ &= \frac{1}{20}\end{aligned}$$

You conclude that $0.050 = \frac{1}{20}$.

11.20 Express the decimal as a fraction: 0.84.

Two digits appear right of the decimal point, terminating in the hundredths place. Those digits comprise the numerator of the equivalent fraction, and the denominator is equal to 100.

$$0.84 = \frac{84}{100}$$

Reduce the fraction to lowest terms.

$$\frac{84}{100} = \frac{84 \div 4}{100 \div 4} = \frac{21}{25}$$

You conclude that $0.84 = \frac{21}{25}$.

11.21 Verify your answer to Problem 11.20 by converting your solution, a fraction, into a decimal.

According to Problem 11.20, $0.84 = \frac{21}{25}$. Verify the fractional representation of 0.84 by calculating the quotient $21 \div 25$. The result should be 0.84.

$$\begin{array}{r} .84 \\ 25\overline{)\ 21.00} \\ \underline{-20\,0} \\ 1\,00 \\ \underline{-\,1\,00} \\ 0 \end{array}$$

11.22 Express the decimal as a fraction: 0.025.

Three digits appear right of the decimal point, terminating in the thousandths place. Thus, the denominator of the fraction should be 1,000. Although one of the decimal digits is 0, it must be counted among the decimal digits. You may omit the 0, however, when you record the decimal digits in the numerator of the fraction.

The only time you don't count 0s as decimal digits are when they appear at the end of the decimal.

$$\frac{025}{1{,}000} = \frac{25}{1{,}000}$$

Reduce the fraction to lowest terms.

$$\frac{25 \div 25}{1{,}000 \div 25} = \frac{1}{40}$$

11.23 Express the decimal as a fraction: 0.348.

Three digits appear right of the decimal point, so the denominator of the equivalent fraction is 1,000; the numerator is the string of decimal digits: 348.

$$0.348 = \frac{348}{1{,}000}$$

Reduce the fraction to lowest terms.

$$\frac{348 \div 4}{1{,}000 \div 4} = \frac{87}{250}$$

11.24 Express the decimal as a mixed number: 1.4.

Separate the decimal into its whole portion 1 and its decimal portion 0.4. The whole portion of the decimal also serves as the whole portion of the equivalent mixed number. To identify the proper fraction portion of the mixed number, express the decimal portion as a fraction.

$$\begin{aligned} 0.4 &= \frac{4}{10} \\ &= \frac{4 \div 2}{10 \div 2} \\ &= \frac{2}{5} \end{aligned}$$

Write the proper fraction portion of the mixed number next to the whole number portion to complete the problem.

$$1.4 = 1\frac{2}{5}$$

11.25 Verify your solution to Problem 11.24 by expressing the mixed number as a decimal.

Multiply the denominator by the whole number, add the numerator, and divide by the denominator.

According to Problem 11.24, $1.4 = 1\frac{2}{5}$. Convert the mixed number into an improper fraction.

$$\begin{aligned} 1\frac{2}{5} &= \frac{5 \cdot 1 + 2}{5} \\ &= \frac{5+2}{5} \\ &= \frac{7}{5} \end{aligned}$$

Verify that the quotient $7 \div 5$ is equal to 1.4.

$$\begin{array}{r} 1.4 \\ 5\overline{)\ 7.0} \\ \underline{-5} \\ 2\,0 \\ \underline{-2\,0} \\ 0 \end{array}$$

11.26 Express the decimal as an improper fraction: 9.55.

Begin by converting the decimal into a mixed number by applying the procedure outlined in Problem 11.24: Separate the decimal into whole and decimal portions (9 and 0.55) and then express the decimal portion as a fraction.

$$\begin{aligned}0.55 &= \frac{55}{100}\\ &= \frac{55 \div 5}{100 \div 5}\\ &= \frac{11}{20}\end{aligned}$$

The mixed number equivalent of 9.55 is the whole portion (9) together with the proper fraction calculated above.

$$9.55 = 9\frac{11}{20}$$

Convert the mixed number into an improper fraction.

$$\begin{aligned}9\frac{11}{20} &= \frac{20 \cdot 9 + 11}{20}\\ &= \frac{180 + 11}{20}\\ &= \frac{191}{20}\end{aligned}$$

You conclude that $9.55 = \frac{191}{20}$.

11.27 Express the decimal as an improper fraction: 3.0026.

Convert the decimal portion into a proper fraction. Note that four digits appear right of the decimal point, terminating in the ten thousandths place.

$$\begin{aligned}0.0026 &= \frac{0026}{10{,}000}\\ &= \frac{26}{10{,}000}\end{aligned}$$

Reduce the fraction to lowest terms.

$$\begin{aligned}&= \frac{26 \div 2}{10{,}000 \div 2}\\ &= \frac{13}{5{,}000}\end{aligned}$$

Combine the proper fraction with the whole portion from the original decimal (3).

$$3.0026 = 3\frac{13}{5{,}000}$$

Convert the mixed number into an improper fraction.

$$= \frac{5{,}000(3)+13}{5{,}000}$$
$$= \frac{15{,}000+13}{5{,}000}$$
$$= \frac{15{,}013}{5{,}000}$$

11.28 Express the decimal as a fraction: $0.\bar{2}$.

To express a repeating decimal as a fraction, apply the following procedure:

1. Count the number of digits in the repeating string, the number of digits below the horizontal bar in the decimal.
2. Construct a fraction whose numerator is a single copy of the repeating string.
3. Complete the fraction with a denominator consisting only of the digit 9, appearing once for each digit in the repeating string.

You counted the digits in Step 1. If you count three digits in that step, there should be three 9s in the denominator (999). Only use the digit 9.

The decimal $0.\bar{2} = 0.2222222...$ consists of the single digit 2 repeating infinitely. That repeating digit is the numerator of the equivalent fraction. Because a single digit repeats in this decimal, the single digit 9 appears in the denominator.

$$0.\bar{2} = \frac{2}{9}$$

11.29 Verify your answer to Problem 11.28 by returning the fraction to decimal form.

Apply long division to calculate the quotient $2 \div 9$ and verify that the result is $0.\bar{2}$.

$$\begin{array}{r} .222 \\ 9\overline{)\ 2.000} \\ \underline{-1\,8} \\ 20 \\ \underline{-18} \\ 20 \\ \underline{-18} \\ 2 \end{array}$$

Consider the sequence of differences calculated above: 2, 2, 2. Placing additional zeroes at the end of the dividend will consistently produce a difference of 2, so you conclude that the quotient is a repeating decimal: $2 \div 9 = 0.\bar{2}$.

11.30 Express the decimal as a fraction: $0.\bar{6}$.

The decimal consists of one repeated digit: 6. Therefore, the numerator of the equivalent fraction is 6 and the denominator is a single 9. Recall that the denominator is a whole number consisting entirely of 9s; the number of 9s is dictated by the number of digits that are repeated in the decimal.

$$0.\bar{6}=\frac{6}{9}$$

Reduce the fraction to lowest terms.

$$\begin{aligned}0.\bar{6}&=\frac{6\div 3}{9\div 3}\\&=\frac{2}{3}\end{aligned}$$

11.31 Express the decimal as a fraction: $0.\overline{97}$.

The decimal $0.\overline{97}$ is constructed of two infinitely repeating digits: $0.\overline{97}=0.979797979797\ldots$. Place the repeating digits in the numerator of the equivalent fraction. The denominator of the fraction is 99—a two-digit number consisting entirely of 9s. The number of 9s in the denominator is equal to the number of digits repeated infinitely in the decimal.

$$0.\overline{97}=\frac{97}{99}$$

Note that this process, first outlined in Problem 11.28, does not apply if any non-repeating digits appear on the right side of the decimal point. The entire decimal portion of the number must consist of repeating digits.

For example, you couldn't use this technique to convert $0.8\overline{21}=0.821212121\ldots$ into a fraction because the tenths digit is not part of the repeating pattern.

11.32 Express the decimal as a mixed number: $5.\overline{186}$.

The whole part of the decimal (5) also serves as the whole part of the mixed number. Notice that the decimal portion of the number is the three-digit, infinitely repeating string 186. A single copy of the repeated number pattern appears in the numerator of the equivalent fraction; because the pattern contains three digits, the denominator should contain three 9s.

$$5.\overline{186}=5\frac{186}{999}$$

Reduce the proper fraction to lowest terms.

$$\begin{aligned}5\frac{186}{999}&=5\frac{186\div 3}{999\div 3}\\&=5\frac{62}{333}\end{aligned}$$

11.33 Identify the decimal equivalent of $\frac{41}{9,999}$ by implementing the process described in Problem 11.28 and verify your answer.

According to the procedure outlined in the solution to Problem 11.28, a denominator of 9,999 indicates a decimal with a four-digit string that infinitely repeats. The numerator should contain the four-digit string of digits. Because the numerator contains only two digits (41), you should insert two zeroes immediately preceding those digits: 41 = 0041.

$$\begin{aligned}\frac{41}{9,999} &= 0.004100410041\ldots\\ &= 0.\overline{0041}\end{aligned}$$

To verify that you have identified the correct decimal, apply long division to ensure that $41 \div 9,999 = 0.\overline{0041}$.

$$\begin{array}{r} .00410041 \\ 9,999\overline{)\ 41.00000000} \\ \underline{-39996} \\ 10040 \\ \underline{-\ 9999} \\ 41000 \\ \underline{-\ 39996} \\ 10040 \\ \underline{-\ 9999} \\ 41 \end{array}$$

You need to drop down two extra zeroes here, so you can divide 41,000 (not 410) by 9,999. That means you need to stick two extra zeroes in the quotient as well. They go right here.

Simplifying Fraction and Decimal Expressions

Work with both at the same time

11.34 Simplify the expression and write the answer in decimal form: $\frac{1}{2} + 0.089$.

The problem directs you to report the sum as a decimal, so apply long division to express the fraction $\frac{1}{2}$ as a decimal.

$$\begin{array}{r} .5 \\ 2\overline{)\ 1.0} \\ \underline{-1\,0} \\ 0 \end{array}$$

Substitute the decimal equivalent calculated above into the original expression.

$$\frac{1}{2} + 0.089 = 0.5 + 0.089$$

Calculate the sum.

$$\begin{array}{r} 0.500 \\ \underline{+0.089} \\ 0.589 \end{array}$$

11.35 Simplify the expression and write the answer as a fraction: $\frac{4}{5}+0.31$.

You are instructed to report the sum as a fraction, so convert the decimal 0.31 into a fraction.

$$0.31=\frac{31}{100}$$

Substitute the equivalent fraction into the original expression.

$$\frac{4}{5}+0.31=\frac{4}{5}+\frac{31}{100}$$

Express the sum in terms of the least common denominator.

$$\begin{aligned} &=\frac{4\cdot 20}{5\cdot 20}+\frac{31}{100} \\ &=\frac{80}{100}+\frac{31}{100} \\ &=\frac{80+31}{100} \\ &=\frac{111}{100} \end{aligned}$$

11.36 Simplify the expression and write the answer as a decimal rounded to the thousandths place: $\frac{10}{11}-0.\overline{47}$.

You are directed to express the difference as a decimal, so apply long division to convert the fraction into decimal form.

$$\begin{array}{r} .9090 \\ 11\overline{)\ 10.0000} \\ \underline{-\ 9\ 9} \\ 100 \\ \underline{-99} \\ 10 \end{array}$$

Just like Problem 11.33, you have to copy an extra zero from the dividend next to the difference (1) so you can divide 11 into 100 instead of dividing it into 10. Make sure to stick an extra zero in the quotient as well.

Substitute the decimal equivalent calculated above into the original expression.

$$\frac{10}{11}-0.\overline{47}=0.\overline{90}-0.\overline{47}$$

You are instructed to calculate the difference and round it to the thousandths place. In order to round to a specific place, you examine the digit one place to the right. In this case, you examine the ten thousandths digit in order to round to the thousandths place. Both of the decimals in the expressions repeat infinitely, but only the digits up to and including the ten thousandths place must be included.

$$\begin{array}{r} 0.9090 \\ -0.4747 \\ \hline 0.4343 \end{array}$$

The ten thousandths digit of the difference (3) is less than 5, so the thousandths digit is not changed when the difference is rounded.

$$\frac{10}{11}-0.\overline{47}=0.434$$

11.37 Simplify the expression and write the answer as a decimal rounded to the thousandths place: $0.3\left(\frac{7}{12}\right)$.

Apply long division to convert the fraction into a decimal.

$$\begin{array}{r} .5833 \\ 12\overline{)\ 7.0000} \\ -6\,0 \\ \hline 1\,00 \\ -96 \\ \hline 40 \\ -36 \\ \hline 40 \\ -36 \\ \hline 4 \end{array}$$

The quotient is an infinitely repeating decimal: $\frac{7}{12}=0.58\overline{3}$. As Problem 11.36 explains, in order to round to the thousandths place, you must consider the digit in the ten thousandths place, so replace the fraction $\frac{7}{12}$ with the decimal 0.5833 and multiply by 0.3.

$$\begin{array}{r} 0.5833 \\ \times\quad 0.3 \\ \hline 1\,7499 \end{array}$$

A total of five digits appear right of the decimal points in the product, so you conclude that $0.5833(0.3) = 0.17499$. Therefore, $0.3\left(\frac{7}{12}\right)$ rounded to the thousandths place is equal to 0.175.

11.38 Repeat Problem 11.37, this time reporting the product $0.3\left(\frac{7}{12}\right)$ as a fraction.

The decimal 0.3 has a single digit 3 that terminates in the tenths place. Thus, the equivalent fraction has a numerator of 3 and a denominator of 10.

$$0.3\left(\frac{7}{12}\right)=\frac{3}{10}\left(\frac{7}{12}\right)$$

The numerator value 3 and the denominator value 12 have common factor 3. Reduce the product by eliminating the common factor before you multiply.

$$\begin{aligned}&=\frac{\overset{1}{\cancel{3}}}{10}\left(\frac{7}{\underset{4}{\cancel{12}}}\right)\\&=\frac{1}{10}\left(\frac{7}{4}\right)\\&=\frac{7}{40}\end{aligned}$$

If you convert this fraction into a decimal, the result is $\frac{7}{40}=0.175$. Recall that Problem 11.37 reported a final answer of $0.174\overline{9}$, which is very nearly equal to 0.175. In fact, once rounded, both had a final answer of 0.175.

Problem 11.38, the answer in fraction form, is exactly accurate, whereas the decimal form of the answer, presented in Problem 11.37, is limited by the number of digits you use in the product. These two problems reinforce an important rule: fractions always provide an exact answer, whereas decimals may only provide answers correct to a specific number of decimal places. Thus, you should elect to report answers as fractions (instead of decimals) whenever an expression contains a fraction.

11.39 Simplify the expression and write the answer as an improper fraction: $\frac{8}{3}\div 1.\overline{1}$.

Begin by converting the decimal into a fraction. According to Problem 11.28, $1.\overline{1}=1\frac{1}{9}$; the decimal portion of $1.\overline{1}$ consists of a single digit that repeats infinitely, so the numerator of the equivalent fraction is the repeated digit (1) and the denominator is the single digit 9.

$$\frac{8}{3}\div 1.\overline{1}=\frac{8}{3}\div 1\frac{1}{9}$$

Express the mixed number as an improper fraction.

$$\begin{aligned}&=\frac{8}{3}\div\frac{9\cdot 1+1}{9}\\&=\frac{8}{3}\div\frac{9+1}{9}\\&=\frac{8}{3}\div\frac{10}{9}\end{aligned}$$

Dividing by a fraction is the equivalent of multiplying by its reciprocal.

Look back at Problem 9.15 if you need to review this.

$$=\frac{8}{3}\cdot\frac{9}{10}$$

Reduce the product before multiplying the fractions. Notice that 8 and 10 share common factor 2; 3 and 9 share common factor 3.

$$\begin{aligned}&=\frac{4\cancel{8}}{1\cancel{3}}\cdot\frac{3\cancel{9}}{5\cancel{10}}\\&=\frac{4}{1}\cdot\frac{3}{5}\\&=\frac{12}{5}\end{aligned}$$

Chapter 12

PERCENTAGES

A reference point for decimals and fractions

In the preceding chapters, you have investigated two significant means by which to report portions of whole numbers: fractions and decimals. This chapter explores percentages, a third method with very clear connections to its predecessors. It begins by exploring the techniques used to convert between fractions, decimals, and percentages and then moves into topics unique to percentages.

The word "percent" really means "per cent" or "out of 100," because "cent-" is a prefix meaning 100. (A century describes 100 years, 100 cents make a dollar, a centennial is a 100-year anniversary, there are 100 centimeters in a meter, etc.) That means the phrase "85%" or "85 percent" translates into "85 out of 100."

Why are percentages useful? They standardize things. I don't know about you, but a test grade of "103 out of 136 correct" is not as easy to interpret as "75.7%."

This chapter explains how to calculate percentages, and it may even help you understand store discounts better, if you've ever wondered how to calculate a 30% discount or what a 210% markup means.

Converting Between Percentages, Fractions, and Decimals

Move the decimal point right or left

12.1 Identify the fraction implied by "15 percent" and explain your answer.

The phrase "15 percent" is commonly written using the percent symbol in mathematics: 15%; it compares the given quantity to a larger population of 100. For instance, in this example, 15% literally means "15 out of 100."

To express a percentage as a fraction, divide it by 100 and reduce the fraction to lowest terms.

$$\begin{aligned} 15\% &= \frac{15}{100} \\ &= \frac{15 \div 5}{100 \div 5} \\ &= \frac{3}{20} \end{aligned}$$

12.2 Express the percentage as a fraction reduced to lowest terms: 88%.

Apply the technique described in Problem 12.1: Construct a fraction with a denominator equal to the number preceding the percent sign and a denominator of 100.

$$88\% = \frac{88}{100}$$

Reduce the fraction to lowest terms, dividing the numerator and denominator by the greatest common factor, 4.

$$\begin{aligned} \frac{88}{100} &= \frac{88 \div 4}{100 \div 4} \\ &= \frac{22}{25} \end{aligned}$$

12.3 Express the percentage as a fraction reduced to lowest terms: 6%.

Divide the value preceding the percent sign by 100 and reduce the fraction to lowest terms, dividing both the numerator and denominator by 2, the greatest common factor.

$$\begin{aligned} 6\% &= \frac{6}{100} \\ &= \frac{6 \div 2}{100 \div 2} \\ &= \frac{3}{50} \end{aligned}$$

12.4 Express $\frac{1}{2}\%$ as a complex fraction and simplify it.

According to the technique described in Problem 12.1, to convert a percentage into a fraction, you should divide the number preceding the percent sign by 100.

$$\frac{1}{2}\% = \frac{1/2}{100}$$

The result is a complex fraction—a fraction whose numerator and/or denominator is also a fraction—and should be simplified. Express the complex fraction as a quotient of the numerator and denominator.

Problem 9.31 lists the steps you should follow to simplify a complex fraction, in case you need to review.

$$= \frac{1}{2} \div 100$$

Divide the integer 100 by 1 to convert it into a fraction. Recall that any number divided by 1 is equal to the original number, so this step does not alter the value of the expression.

$$= \frac{1}{2} \div \frac{100}{1}$$

Convert the quotient into a product by replacing the divisor with its reciprocal.

$$\begin{aligned} &= \frac{1}{2} \cdot \frac{1}{100} \\ &= \frac{1}{2 \cdot 100} \\ &= \frac{1}{200} \end{aligned}$$

This makes sense. If there are 100 ones in 100, then there should be 200 one-halves in 100.

Therefore, $\frac{1}{2}\% = \frac{1}{200}$.

12.5 Express $7\frac{3}{5}\%$ as a complex fraction and simplify it.

Express the mixed number as an improper fraction.

$$\begin{aligned} 7\frac{3}{5} &= \frac{5 \cdot 7 + 3}{5} \\ &= \frac{35 + 3}{5} \\ &= \frac{38}{5} \end{aligned}$$

Thus, $7\frac{3}{5}\% = \frac{38}{5}\%$. Apply the procedure described in Problem 12.4 to convert the new percentage into a fraction.

$$\begin{aligned}\frac{38}{5}\% &= \frac{38/5}{100}\\ &= \frac{38}{5} \div 100\\ &= \frac{38}{5} \div \frac{100}{1}\\ &= \frac{38}{5} \cdot \frac{1}{100}\end{aligned}$$

Reduce the product before multiplying.

$$\begin{aligned}&= \frac{\overset{19}{\cancel{38}}}{5} \cdot \frac{1}{\underset{50}{\cancel{100}}}\\ &= \frac{19}{5} \cdot \frac{1}{50}\\ &= \frac{19 \cdot 1}{5 \cdot 50}\\ &= \frac{19}{250}\end{aligned}$$

You conclude that $7\frac{3}{5}\% = \frac{19}{250}$.

12.6 Express the decimal as a percentage: 0.97.

To convert a decimal into a percentage, move the decimal point two places to the right. In this problem, you should move the decimal to the right of the tenths place and then to the right of the hundredths place. Once the decimal is correctly positioned, place a percent sign at the right side of the number.

$$0.97 = 97\%$$

12.7 Express the decimal as a percentage: 2.37.

As Problem 12.6 explains, to convert a decimal into a percentage, move the decimal point two places to the right and end the number with a percent sign.

$$2.37 = 237\%$$

12.8 Express the decimal as a percentage: 0.015.

When you move the decimal point two places to the right and add a percent sign, the resulting value still contains a decimal point.

$$0.015 = 001.5\%$$

Zeroes on the left side of a decimal point that begin a number, like zeroes on the right side of a decimal point that end a number, should be omitted from the final answer.

$$0.015 = 1.5\%$$

12.9 Express the percentage as a decimal: 64.3%.

According to Problem 12.6, to convert a decimal into a percentage, you move the decimal point two places to the right. Conversely, to convert from a percentage to a decimal, you move the decimal point two places to the left.

$$64.3\% = 0.643$$

RULE OF THUMB: The percentage version of a number always looks bigger. For example, 0.5 = 50%, and 0.5 is less than 50. That's a good way to make sure you moved the decimal in the right direction.

12.10 Express the percentage as a decimal: 110%.

As Problem 12.9 explains, you convert a percentage into a decimal by moving the decimal point in the percentage two places to the left. In this problem, you are given a whole number that does not contain an explicitly stated decimal point. Place a decimal point at the end of the whole number, as demonstrated below.

$$110\% = 110.0\%$$

Add a zero in the tenths place for context—to make it clear that the punctuation mark is a decimal point and not an oddly positioned period.

Move the decimal point two places to the left.

$$110.0\% = 1.100$$

Omit the zeroes at the end of the decimal.

$$1.100 = 1.1$$

12.11 Express the percentage as a decimal: 0.092%.

Move the decimal point two places to the left.

$$0.092\% = 0.00092$$

12.12 Express the fraction as a percentage: $\frac{7}{20}$.

To convert a fraction into a percentage, first convert it to a decimal and then apply the technique described in Problems 12.6–12.8: Move the decimal two places to the right to convert the decimal into a percentage.

Begin by applying long division to calculate the decimal equivalent of the fraction.

$$\begin{array}{r} .35 \\ 20\overline{)\ 7.00} \\ -6\,0 \\ \hline 1\,00 \\ -1\,00 \\ \hline 0 \end{array}$$

Thus, $\frac{7}{20} = 0.35$. Convert the decimal into a percentage to complete the problem.

$$0.35 = 35\%$$

12.13 Express the fraction as a percentage: $\frac{9}{16}$.

Apply long division to convert the fraction into a decimal.

$$\begin{array}{r} .5625 \\ 16\overline{)\;9.0000} \\ -8\,0 \\ \hline 1\,00 \\ -96 \\ \hline 40 \\ -32 \\ \hline 80 \\ -80 \\ \hline 0 \end{array}$$

Express the decimal as a percentage by moving the decimal point two places to the right.

$$0.5625 = 56.25\%$$

12.14 Express the fraction as a percentage: $\frac{17}{100{,}000}$.

The steps are listed in the solution to Problem 11.18.

Chapter 11 describes the technique used to convert decimals into fractions. Of particular interest in this problem is the method for converting *terminating* decimals into fractions. When the denominator of a fraction is a power of 10 (examples include 10; 100; 1,000; 10,000), it is a simple matter to express that fraction as a decimal by reversing the method presented in the preceding chapter.

The denominator of this fraction is a power of 10; the fraction is read "seventeen one hundred thousandths," so the equivalent fraction should have five digits to the right of the decimal point and terminate in the one hundred thousandths place. You may also choose to count the number of zeroes in a denominator that is a power of 10. The number of zeroes is equal to the number of decimal digits in the equivalent decimal—in this case, five.

The numerator of the fraction represents the digits that are to appear right of the decimal point. In this problem, the numerator contains only two digits, so you must insert three place holder zeroes immediately after the decimal point to ensure that 1 and 7 are the fourth and fifth digits, respectively, following the decimal point.

$$\frac{17}{100{,}000} = 0.00017$$

To convert the decimal into a percentage, move the decimal point two places to the right.

$$0.00017 = 000.017\%$$

Omit unnecessary zeroes in the final answer.

$$\frac{17}{100{,}000} = 0.017\%$$

12.15 Express the fraction as a percentage: $\frac{23}{3}$.

The same technique is applied to convert both improper and proper fractions into decimals; use long division to divide the numerator by the denominator, continuing to place zeroes after the decimal point in the dividend until the quotient either terminates or infinitely repeats.

$$\begin{array}{r} 7.66 \\ 3\overline{)\ 23.00} \\ \underline{-21} \\ 2\,0 \\ \underline{-1\,8} \\ 20 \\ \underline{-\ 18} \\ 2 \end{array}$$

Thus, $\frac{23}{3} = 7.\overline{6}$, or 7.66666.... To express the decimal as a percentage, move the decimal point two places to the right.

$$\frac{23}{3} = 766.6666...\%$$

The decimal portion of the number consists of the infinitely repeating single digit 6. Apply shorthand notation to identify the repeating digit, as demonstrated below.

$$\frac{23}{3} = 766.\overline{6}\%$$

12.16 Express the fraction as a percentage: $\frac{14}{99}$.

Problem 11.28 summarizes a technique used to convert infinitely repeating decimals into fractions. According to the steps listed in that solution, the repeating string of digits should appear as the numerator of a fraction, and the denominator should consist entirely of 9s. Furthermore, the number of 9s appearing in the denominator is equal to the number of digits that are repeated in the decimal.

In this problem, you are given the fraction $\frac{14}{99}$. Its denominator contains two 9s, so the decimal representation of the fraction infinitely repeats the digits 14. Apply long division to verify this observation.

$$\begin{array}{r} .1414 \\ 99\overline{)14.0000} \\ \underline{-9\,9} \\ 4\,10 \\ \underline{-3\,96} \\ 140 \\ \underline{-\,99} \\ 410 \\ \underline{-\,396} \\ 14 \end{array}$$

Thus, $\frac{14}{99} = 0.141414\ldots$. Convert the decimal into a percentage by moving the decimal point two places to the right. Use shorthand notation to identify the infinitely repeating decimal values.

$$\frac{14}{99} = 14.\overline{14}\%$$

Percentage Increase and Decrease

How much of the original amount changed?

12.17 An item with a suggested retail price of $12 is discounted 40% at a local store. Calculate the discounted price of the item.

To calculate percentage increase and decrease, apply the formula $s \cdot p = c$, in which s represents the starting value, p represents the percentage change (expressed as a decimal), and c represents the change from the original value. Most percentage change problems will provide two of the three values and you are required to calculate the third.

C answers the question, "What is P percent of S?" In this problem, C represents 40% of $12, the amount of the discount.

In this problem, you are given a suggested price of $12, so $s = 12$. You are also given the percentage change: $p = 40\%$. Convert the percentage into a decimal by moving the decimal point two places to the left.

$$p = 0.40$$

Substitute the known values of s and p into the percentage change formula.

$$\begin{aligned} s \cdot p &= c \\ 12(0.40) &= c \end{aligned}$$

According to the above equation, $c = 12(0.40)$. Multiply 12 and 0.40 to calculate c.

$$\begin{array}{r} 12 \\ \times\ .40 \\ \hline 4.80 \end{array}$$

Note that $c = \$4.80$ is not the new price of the item. Rather, c represents 40% of 12. In other words, the original price of \$12—when discounted 40%—will decrease by \$4.80. Subtract c from s to calculate the new price.

$$\begin{array}{r} 12.00 \\ -4.80 \\ \hline 7.20 \end{array}$$

The discounted price of the item is \$7.20.

12.18 The state sales tax in Maryland is 6%. Calculate the total cost of a taxable item with a pre-tax purchase price of \$300.

Sales tax is a fee added to the purchase price of an item. In this problem, the consumer is responsible for the purchase price of \$300, as well as an additional 6% of that purchase price. The starting purchase price is \$300, so $s = 300$; the percentage value is $p = 6\% = 0.06$. Substitute these values into the percentage change formula.

$$\begin{aligned} s \cdot p &= c \\ 300(0.06) &= c \end{aligned}$$

Calculate the product to determine the value of c. Recall that c is equal to p percent of s, in this case, 6% of \$300.

$$\begin{array}{r} 300 \\ \times\ .06 \\ \hline 18.00 \end{array}$$

Thus, 6% of \$300 is \$18. Specifically, the tax on the purchase is $c = 18$. Add the sales tax to the pre-tax price to calculate the total cost of the item.

$$\$300 + \$18 = \$318$$

12.19 At the end of 2009, the population of a small town was 2,500 people. It is determined that the population decreased by 9% during 2010. Calculate the population of the town at the end of 2010.

This problem identifies the original values of the population and the percentage by which it decreases: $s = 2{,}500$ and $p = 9\% = 0.09$, respectively. Substitute these values into the percentage change formula to calculate c.

When you take P% of S, you get C. In this case, you'll calculate 9% of 2,500, which is the number of people that left during the year.

$$\begin{aligned} s \cdot p &= c \\ 2{,}500(0.09) &= c \\ 225 &= c \end{aligned}$$

The population of the small town decreased by a total of $c = 225$ people between January 1, 2010 and December 31, 2010. Therefore, the population at the end of 2010 was $2{,}500 - 225 = 2{,}275$.

12.20 After redistricting, a school population grows from 2,150 to 2,875. Calculate the percentage increase in the population, accurate to the tenths place.

This problem provides the starting value $s = 2{,}150$ and the final value, once the change has been applied. You are not given the actual change value c, the amount by which the original value either increased or decreased; however, calculating c is a simple matter. To determine the total change, subtract the original value from the final, larger value.

$$c = 2{,}875 - 2{,}150 = 725$$

Now that you have identified s and c, substitute them into the percentage change equation.

$$s \cdot p = c$$
$$2{,}150 \cdot p = 725$$

You want it to look like "p =" just like Problems 12.17–12.19 looked like "c =" once you got to this step.

Notice that p is multiplied by a number on the left side of the equation. To complete this problem, you need to isolate p by itself on the left side of the equal sign. To accomplish this, divide the value on the other side of the equation (725) by the number next to p.

$$p = \frac{725}{2{,}150}$$

Apply long division to calculate the quotient $725 \div 2{,}150$ or—if allowed by your instructor—use a calculator.

$$p \approx 0.337209302$$

Convert the decimal into a percentage.

$$p \approx 33.7209302\%$$

The problem directs you to round to the tenths place.

$$p = 33.7\%$$

The population of the school increased by 33.7% after redistricting.

"in³" is read "cubic inches." You can fit 25 cubes of helium into the balloon, if each little gassy cube had a length, width, and height of one inch.

12.21 A latex balloon purchased at a florist contains 25 in^3 of helium at the time of purchase. Twenty-four hours later, the balloon contains only 21 in^3 of helium. Calculate the percentage decrease in volume.

The problem identifies the starting value $s = 25$ and directs you to calculate the percentage change. Before you can calculate p, you must first calculate c, the amount by which the starting value changed after 24 hours.

$$c = 25 - 21 = 4$$

Substitute $s = 25$ and $c = 4$ into the percentage change equation.

$$s \cdot p = c$$
$$25 \cdot p = 4$$

Notice that p is not isolated on one side of the equal sign, because it is multiplied by 25. Divide by 25 to solve the equation and calculate p.

$$\begin{aligned} p &= \frac{4}{25} \\ &= 0.16 \end{aligned}$$

Convert the decimal into a percentage to complete the problem. Over a 24-hour period, the balloon's volume decreased 16%.

12.22 A total of 44,200 fans attend a football game, which occupies 85% of the stadium's available seats. Calculate the total seating capacity of the stadium.

This problem states that 85% of the total seating capacity of the stadium is 44,200. You are given $p = 0.85$ and $c = 44{,}200$ and need to calculate s. If you have difficulty deciding which values should be assigned to which variables, remember that c should always be equal to p% of s. In this problem, $c = 44{,}200$ is equal to $p = 85\%$ of the unknown value s.

Before you calculate s, think about what it should look like. It has to be bigger than 44,200, because 15% of the seats weren't full when the crowd totaled 44,200.

$$\begin{aligned} s \cdot p &= c \\ s(0.85) &= 44{,}200 \end{aligned}$$

Notice that s is not isolated on one side of the equation—it is multiplied by 0.85. Divide 44,200 by that value to solve for s.

$$s = \frac{44{,}200}{0.85}$$

Apply long division to calculate s. Remember that no digits of the divisor may appear right of the decimal point. Therefore, you should move the decimal point two places to the right in the divisor and the dividend. This adds two additional zeroes to the dividend.

$$\begin{array}{r} 52000 \\ 85\overline{)\ 44{,}20000} \\ \underline{-42\,5} \\ 1\,70 \\ \underline{-1\,70} \\ 0 \end{array}$$

You conclude that the stadium has a maximum capacity of $s = 52{,}000$ fans.

12.23 A company purchases a large cake for its holiday party. After one hour, approximately half of the cake remains. After another hour elapses, half of the remaining cake is eaten.

This trend continues for a total of five hours—at the end of each hour, approximately half of the cake that was available at the beginning of the hour has been eaten. Calculate the percentage of the original cake that remains at the end of the five-hour party.

At the beginning of the party, the whole cake is uneaten; after one hour, half of the cake remains. Recall that the word "of," especially when used in reference to fractions, indicates multiplication. Multiply one (which represents one, whole cake) by one-half (the amount of cake remaining) to identify the portion of cake remaining at the end of the first hour.

$$1\cdot\frac{1}{2}=\frac{1}{2}$$

Multiply that product again by one-half to identify the amount of cake remaining at the end of the second hour.

$$\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$$

Repeat this process three more times, multiplying the preceding product by one-half in each case.

$$\frac{1}{4}\cdot\frac{1}{2}=\frac{1}{8}$$
$$\frac{1}{8}\cdot\frac{1}{2}=\frac{1}{16}$$
$$\frac{1}{16}\cdot\frac{1}{2}=\frac{1}{32}$$

At the end of five hours, approximately $\frac{1}{32}$ of the original cake remains. Apply long division to convert the fraction into a decimal.

$$\begin{array}{r} .03125 \\ 32\overline{)1.00000} \\ \underline{-96} \\ 40 \\ \underline{-32} \\ 80 \\ \underline{-64} \\ 160 \\ \underline{-160} \\ 0 \end{array}$$

Move the decimal point two places to the right to express the decimal as a percentage.

$$0.03125 = 3.125\%$$

At the end of the five-hour party, only 3.125% of the cake remains.

Note: In Problems 12.24–12.25, a consumer is comparing the sale prices of an item at two different stores. The manufacturer's suggested retail price for the item is $800. Assume no sales tax is applied to the purchase.

12.24 Calculate the sale price of the item at Store A, which offers a 30% discount.

The suggested retail price of the item is $s = 800$, and Store A offers a $p = 0.30 = 30\%$ discount. Substitute these values into the percentage change equation to calculate c, the total discount (in dollars) offered by Store A.

$$\begin{aligned} s \cdot p &= c \\ 800(0.30) &= c \\ 240 &= c \end{aligned}$$

Store A offers the item at a $c = \$240$ discount; calculate the discounted price of the item by subtracting c from s.

$$s - c = 800 - 240 = 560$$

The sale price of the item at Store A is $560.

Note: In Problems 12.24–12.25, a consumer is comparing the sale prices of an item at two different stores. The manufacturer's suggested retail price for the item is $800. Assume no sales tax is applied to the purchase.

12.25 Store B offers a similar sale, discounting the suggested price 15% and then applying a second discount of 15% to the already discounted price. The sale circular claims a total of 15% + 15% = 30% in savings. Support or dispute this claim and explain your answer.

The sale at Store B is a bit more complicated than the sale at Store A, described in Problem 12.24. Once again, the suggested retail price is $s = 800$, but the discount $p = 0.15 = 15\%$ is applied twice in succession. Use the percentage change equation to calculate the initial discount of 15%.

$$\begin{aligned} s \cdot p &= c \\ 800(0.15) &= c \\ 120 &= c \end{aligned}$$

Subtract the discount, $c = \$120$, from $s = \$800$ to calculate the price of the item after the first discount.

$$s - c = 800 - 120 = 680$$

The second 15% discount is applied to the new price of $680. Use the percentage change equation once more, this time given $s = 680$ and $p = 0.15$.

$$\begin{aligned} s \cdot p &= c \\ 680(0.15) &= c \\ 102 &= c \end{aligned}$$

Now you know when stores say "25% off regular price and then an additional 25% off at the register," that's not the same as 50% off the regular price. You're onto their trickery.

The second discount is less than the first, because it is calculated based on a smaller initial price—the price that was discounted in the first part of the problem. Calculate the final cost of the item, reflecting both discounts.

$$\$680 - \$102 = \$578$$

The sale price of the item at Store B is $578, which is $18 higher than the $560 sale price at Store A. Therefore, discounting an item 15% twice is not the same as discounting it once at 30%.

Computing Percentages

What percent of something equals something else?

12.26 Use cross-multiplication to demonstrate that the statement below is true.

$$\frac{2}{5} = \frac{4}{10}$$

An equation stating that two fractions are equal is called a proportion. In this case, the equation states that two out of five things is proportionately equal to four out of ten things. A technique called cross-multiplication is used to solve proportions. Its name is derived from the direction in which you multiply.

Consider the diagram below, in which the numerator of the left fraction is multiplied by the denominator of the right fraction. Similarly, the denominator of the left fraction is multiplied by the numerator of the right fraction. Line segments connect the values you multiply, for the purposes of illustration. Because those line segments cross, the technique is termed "cross-multiplication."

$$\frac{2}{5} \times \frac{4}{10}$$
$$2 \cdot 10 = 5 \cdot 4$$
$$20 = 20$$

Notice that the two products are both equal to 20. This result is not unique to this proportion. Cross-multiplying allows you to transform a proportion into another valid equation that does not include fractions.

You'll deal with proportions in a lot more detail in Chapter 17. If you're struggling with Problems 12.27–12.41, skip ahead to Problems 17.28–17.34 to get some help solving proportions.

12.27 Solve the proportion for x: $\frac{x}{3} = \frac{16}{24}$.

This problem contains a variable x, a symbol used to represent an unknown quantity. You're asked to solve for x, which means you should isolate x on one side of the equation and thereby identify that value of that unknown quantity. Because this book has not yet explored solving equations in detail, we will limit the scope of equation solving to proportions containing one unknown.

Cross-multiply, as directed by Problem 12.26, to rewrite the equation. Note that one side of the equation is equal to a product of two numbers, whereas the other side is equal to the product of a number and x.

$$x \cdot 24 = 3 \cdot 16$$

It is customary to express the product $x \cdot 24$ as $24x$; the number usually precedes the variable, and no multiplication dot is needed to indicate multiplication.

$$24x = 3 \cdot 16$$
$$24x = 48$$

To solve the equation for x, divide 48 by 24.

$$x = \frac{48}{24}$$

Technically, you're dividing both sides of the equation by 24, not just the right side. You'll learn more about this later, in Chapter 17. For now, just divide the number NOT next to x by the number that IS next to x.

Divide the numerator and denominator by the greatest common factor (24) to reduce the fraction to lowest terms.

$$\begin{aligned} x &= \frac{48 \div 24}{24 \div 24} \\ &= \frac{2}{1} \\ &= 2 \end{aligned}$$

Therefore, $x = 2$. If you substitute $x = 2$ into the original proportion, you conclude two-thirds is proportionately equal to sixteen twenty-fourths.

$$\frac{2}{3} = \frac{16}{24}$$

12.28 Solve the proportion for x: $\frac{12}{x} = \frac{36}{39}$.

Cross-multiply to transform the proportion into an equation that does not contain fractions.

$$12 \cdot 39 = x \cdot 36$$
$$468 = 36x$$

Divide the number isolated on the left side of the equation by the number that is multiplied by x.

$$\frac{468}{36} = x$$

Reduce the fraction to lowest terms.

$$\begin{aligned} \frac{468 \div 36}{36 \div 36} &= x \\ \frac{13}{1} &= x \\ 13 &= x \end{aligned}$$

Thus, $x = 13$.

12.29 Solve the proportion for x: $\frac{3}{8}=\frac{x}{22}$.

Cross-multiply and solve the equation for x, reducing the fraction to lowest terms.

$$\begin{aligned}3\cdot 22&=8\cdot x\\66&=8x\\\frac{66}{8}&=x\\\frac{66\div 2}{8\div 2}&=x\\\frac{33}{4}&=x\end{aligned}$$

12.30 State the percentage proportion formula and identify the three variables within it.

In other words, use $p = 60$ to represent 60%, NOT $p = 0.60$.

The percentage proportion formula, like the percentage change equation presented in Problem 12.17, allows you to calculate one value given two other values. The following variables are used in the formula:

- w, which represents the whole—the number of which you are calculating a percentage
- p, which represents a portion of w
- c, the percentage written as a whole number, not as a decimal

The formula, written below, is a proportion stating that one fraction is equal to another fraction with a denominator of 100. In other words, the formula equates any fraction with its percentage equivalent.

$$\frac{c}{100}=\frac{p}{w}$$

To help you correctly assign values to the variables in the formula, remember that p should represent c percent of w. For example, consider the following proportion, in which $c = 50$, $p = 2$, and $w = 4$.

$$\frac{50}{100}=\frac{2}{4}$$

The percentage proportion formula above correctly states that 2 is 50% of 4.

When you calculate less than 100% of a number, you'll get something smaller. For example, 50% of a number is equal to half of that number. If $c > 100$, then the opposite is true: $p > w$.

12.31 Use the percentage proportion formula to calculate 25% of 84.

Apply the percentage proportion formula (presented in Problem 12.30) to calculate p, which represents $c = 25$ percent of $w = 84$. To ensure you have correctly assigned the variables, note that $p < w$ when $c < 100$. Substitute c and w into the formula.

$$\frac{c}{100} = \frac{p}{w}$$
$$\frac{25}{100} = \frac{p}{84}$$

Cross-multiply to transform the proportion into an equation that does not contain fractions.

$$25 \cdot 84 = 100 \cdot p$$
$$2{,}100 = 100p$$

To solve the equation for p, divide the number that is isolated on one side of the equation by the number that is multiplied by p.

$$\frac{2{,}100}{100} = p$$
$$21 = p$$

You conclude that 25% of 84 is equal to 21.

Problems 12.32–12.33 demonstrate two different ways to calculate 10% of a number.

12.32 Calculate 10% of 139 by moving a decimal point.

To quickly and accurately calculate 10% of a number, move its decimal point one place to the left. In this problem, you are asked to calculate 10% of 139, which can also be written with an explicitly stated decimal point: 139.0. Move the decimal point one place to the left: 13.90 = 13.9.

This shortcut is valid because calculating 10% of a number is equivalent to dividing that number by 10. (Note that the same is not true for all percentages; for example, you do not divide a number by 50 to calculate 50% of it.) Much like multiplying by 10 adds a zero to that number and effectively moves the decimal point one place to the right, dividing by ten moves the decimal point one place to the left.

Problems 12.32–12.33 demonstrate two different ways to calculate 10% of a number.

12.33 Verify your answer to Problem 12.32 by applying the percentage proportion formula.

To calculate $c = 10$ percent of $w = 139$, substitute the values into the percentage proportion formula and solve for p.

$$\frac{c}{100} = \frac{p}{w}$$
$$\frac{10}{100} = \frac{p}{139}$$

Cross-multiply and solve for p.

$$\begin{aligned}10\cdot 139 &= 100\cdot p\\ 1{,}390 &= 100p\\ \frac{1{,}390}{100} &= p\\ 13.9 &= p\end{aligned}$$

12.34 It is customary to leave a tip of 15% for good service at a restaurant, but some customers struggle to calculate 15% of their bill without the aid of a calculator. Modify the technique described in Problem 12.32 to compute a 15% tip for a bill of $70.

According to Problem 12.32, you can calculate 10% of a number by moving its decimal point one place to the left. Therefore, 10% of $70.00 is equal to $7.00. Once you have calculated 10% of a number, calculating 5% of the same number is a simple matter.

Because $10\% \div 2 = 5\%$, it follows that $\$7.00 \div 2 = \$3.50$ is equal to 5% of the total bill. The sum of these percentages represents 15% of the total bill, as illustrated below.

$$\begin{aligned}15\%\text{ of }\$70 &= (10\%\text{ of }\$70)+(5\%\text{ of }\$70)\\ &= \$7.00+\$3.50\\ &= \$10.50\end{aligned}$$

12.35 Calculate 65% of 112.

You are directed to calculate p, which represents $c = 65$ percent of $w = 112$. Apply the percentage proportion formula and solve for p.

$$\begin{aligned}\frac{c}{100} &= \frac{p}{w}\\ \frac{65}{100} &= \frac{p}{112}\\ 65\cdot 112 &= 100\cdot p\\ 7{,}280 &= 100p\\ \frac{7{,}280}{100} &= p\\ 72.8 &= p\end{aligned}$$

Problem 12.32 tells you to move the decimal point one place to the left to divide by 10. To divide a number by 100, move the decimal point TWO places to the left: $7{,}280 \div 100 = 72.80$. No need to break out the calculator.

12.36 Thirty-six percent of what number is equal to 162?

This problem differs slightly from the problems that precede it, because you are given values that correspond to different variables in the percentage proportion formula. Notice that 162 is equal to 36% of some unknown number. You are not asked to calculate 36% of 162; rather, $p = 162$ is $c = 36\%$ of some larger value w. Apply the percentage proportion formula.

$$\frac{c}{100} = \frac{p}{w}$$
$$\frac{36}{100} = \frac{162}{w}$$

Cross-multiply and solve for w.

$$36 \cdot w = 100 \cdot 162$$
$$36w = 16,200$$
$$w = \frac{16,200}{36}$$
$$w = 450$$

Therefore, 36% of 450 is equal to 162.

To double check your answer, multiply 450 by 0.36 and verify that you get 162.

12.37 Twenty-four is 60% of what number?

You are asked to identify the number w, of which $c = 60$ percent is $p = 24$. Apply the percentage proportion formula.

$$\frac{c}{100} = \frac{p}{w}$$
$$\frac{60}{100} = \frac{24}{w}$$
$$60 \cdot w = 100 \cdot 24$$
$$60w = 2,400$$
$$w = \frac{2,400}{60}$$
$$w = 40$$

You conclude that 24 is 60% of 40.

12.38 Twenty-two percent of what number is equal to 12? Express your answer as a decimal.

Given that $c = 22$ percent of some number w is equal to $p = 12$, you are asked to calculate w.

$$\begin{aligned}\frac{c}{100} &= \frac{p}{w}\\ \frac{22}{100} &= \frac{12}{w}\\ 22\cdot w &= 100\cdot 12\\ 22w &= 1{,}200\\ w &= \frac{1{,}200}{22}\end{aligned}$$

Reduce the fraction to lowest terms.

$$\begin{aligned}w &= \frac{1{,}200\div 2}{22\div 2}\\ w &= \frac{600}{11}\end{aligned}$$

Apply long division to express the fraction as a decimal.

$$\begin{array}{r} 54.545\\ 11\overline{)\;600.000}\\ \underline{-55}\\ 50\\ \underline{-44}\\ 60\\ \underline{-55}\\ 50\\ \underline{-44}\\ 60\\ \underline{-55}\\ 5\end{array}$$

The sequence of differences in the long division problem repeats infinitely (5, 6, 5, 6, 5), so the digits of the quotient repeat infinitely as well (5, 4, 5, 4, 5). You conclude that 22% of $54.\overline{54}$ is 12.

12.39 Thirteen is equal to what percent of 30? Express your answer as a decimal.

In this problem, you are given a number and a portion of that number, and you are asked to identify the percentage that the portion represents. In other words, calculate c given $w = 30$ and $p = 13$.

$$\begin{aligned}
\frac{c}{100} &= \frac{13}{30}\\
c\cdot 30 &= 100\cdot 13\\
30c &= 1{,}300\\
c &= \frac{1{,}300}{30}\\
c &= \frac{1{,}300\div 10}{30\div 10}\\
c &= \frac{130}{3}
\end{aligned}$$

Apply long division to express c as a decimal.

$$\begin{array}{r}
43.33\\
3\overline{)\ 130.00}\\
\underline{-12}\\
10\\
\underline{-9}\\
1\,0\\
\underline{-\ 9}\\
10\\
\underline{-\ 9}\\
1
\end{array}$$

You conclude that 13 is $43.\bar{3}\%$ of 30.

12.40 Eighty is equal to what percent of 16?

Like Problem 12.39, you are given values for p and w and are asked to calculate c. However, unlike Problem 12.39, the portion is greater than the whole: $p = 80$ is greater than $w = 16$. The percentage c will be larger than 100.

For example, 50 is equal to 200% of 25, because 50 is twice as large as 25.

$$\begin{aligned}
\frac{c}{100} &= \frac{p}{w}\\
\frac{c}{100} &= \frac{80}{16}\\
c\cdot 16 &= 100\cdot 80\\
16c &= 8{,}000\\
c &= 500
\end{aligned}$$

Therefore, 80 is 500% of 16, because $16 \cdot 5 = 80$.

12.41 Calculate the percentage of questions a student answered *correctly* on a 72-question test if the student answered 27 of the questions incorrectly. Express your answer as a decimal.

If a student answers 27 of 72 questions incorrectly, then he answers $72 - 27 = 45$ questions correctly. Calculate the percentage c given $p = 45$ of the $w = 72$ answers were correct.

$$\begin{aligned}\frac{c}{100} &= \frac{45}{72}\\ c\cdot 72 &= 100\cdot 45\\ 72c &= 4{,}500\\ c &= \frac{4{,}500}{72}\end{aligned}$$

Reduce the fraction to lowest terms, dividing the numerator and denominator by the greatest common factor, 36.

$$\begin{aligned}c &= \frac{4{,}500 \div 36}{72 \div 36}\\ c &= \frac{125}{2}\end{aligned}$$

Apply long division to express c as a decimal.

$$\begin{array}{r} 62.5 \\ 2\overline{)\ 125.0} \\ \underline{-12} \\ 05 \\ \underline{-\ 4} \\ 1\ 0 \\ \underline{-\ 1\ 0} \\ 0 \end{array}$$

You conclude that the student answered 62.5% of the questions correctly.

Chapter 13
EXPONENTS

An in-depth look at tiny, floating numbers

Exponents are more than a step within the order of operations. In the preceding chapters, you have encountered exponents primarily as notation for repetitive multiplication, a shortcut allowing you to express $4 \cdot 4 \cdot 4$ as the more concise expression 4^3. In this chapter, you will explore additional uses of exponents, including techniques for simplifying exponential expressions. The chapter concludes with the introduction of negative exponents, which represent reciprocals within an expression.

It's time to move away from the weary world of fractions, decimals, and percentages. Don't get me wrong—it's important to know how to add, subtract, multiply, and divide all of those things, and it's just as important to be able to convert from one to another. It just gets a bit tedious after a while.

In this chapter, you'll still work with decimals, but not like you had to in Chapters 10 and 11. Instead, you'll work with really gigantic and really teeny numbers, expressing them in scientific notation. Then you'll move on to some handy exponential tricks like "a power to a power equals the product of the powers." Finally, you'll take a look at negative exponents, which are moody and negative because they're sitting somewhere they don't belong. Intrigued?

Scientific Notation

Something point something times 10 to the something

13.1 Describe the purpose of scientific notation.

Scientific notation is used to express numbers of very large and very small magnitude, including great distances and exact, microscopic measurements. For example, you would use scientific notation both to measure the distances between stars in a galaxy and the distances between genes within a cell.

You can have any number of digits to the right of the decimal point.

13.2 Identify the two elements that together comprise a number in scientific notation.

A number in scientific notation is a product of two specific things:

- A decimal with *a single non-zero digit* left of the decimal point, in the ones place
- The number 10 raised to an exponential power

If you are using scientific notation to represent a number greater than or equal to 10, the exponential power will be positive; when numbers less than zero are written in scientific notation, the exponent is negative. Note that scientific notation is not traditionally used to represent numbers between 0 and 10, as traditional decimals suffice.

13.3 Identify the equivalent whole number: 10^5.

An exponential expression indicates repeated multiplication. Hence, 10^5 is equal to the product of five 10s multiplied together. Work from left to right, multiplying two numbers at a time.

For example, if $n = 3$ then $10^3 = 1{,}000$; if $n = 9$ then $10^9 = 1{,}000{,}000{,}000$.

$$\begin{aligned} 10^5 &= 10\cdot10\cdot10\cdot10\cdot10 \\ &= 100\cdot10\cdot10\cdot10 \\ &= 1{,}000\cdot10\cdot10 \\ &= 10{,}000\cdot10 \\ &= 100{,}000 \end{aligned}$$

Notice that there are five zeroes in 100,000, which is the same number that appears in the exponent. This is no coincidence. You can conclude that for any positive whole number n, the exponential expression 10^n is equal to a number beginning with the digit one followed by n zeroes.

13.4 Identify the equivalent decimal: 10^{-8}.

In other words, write 10^{-8} without using any exponents.

Like Problem 13.3, raising 10 to a negative power will also produce a number containing ones and zeroes. The more negative the exponent, the smaller the resulting number will be. In other words, $10^{-9} < 10^{-5}$ because $-9 < -5$.

To express 10 to a negative power as a decimal, place a decimal point and a string of zeroes to the right of the decimal point. The number of zeroes should be equal to one less than the exponent (if the negative sign in the exponent is ignored). In this problem, you should begin with a decimal followed by seven zeroes, as seven is one less than the exponent eight.

$$0.0000000$$

The final digit in the decimal, which follows the zeroes you listed in the previous step, is 1.

$$10^{-8} = 0.00000001$$

The exponential expression 10^{-8} is equal to a decimal constructed entirely of 0 digits, except the final digit 1, which appears eight places right of the decimal point.

13.5 Express the number in scientific notation: 1,652.

Recall that a number in scientific notation must have exactly one digit positioned left of the decimal point. All four digits of this number currently appear left of the decimal point. You may wish to write the number with a decimal point and a zero in the tenths place to clarify the position of the decimal point.

$$1{,}652.0$$

Move the decimal point to the left, counting the number of digits the decimal point passes. Stop when exactly one non-zero digit appears left of the decimal point, as illustrated below.

Number	*Total digits passed by the decimal point*
1,652.0	0
165.20	1
16.520	2
1.6520	3

If you'd moved the decimal point three places to the RIGHT, the power would have been -3 instead of +3.

When you move the decimal point three digits to the left, the number 1,652.0 becomes 1.6520; the single digit 1 appears left of the decimal point, as required by scientific notation. Because you moved the decimal point three places to the left, raise 10 to the third power.

$$1{,}652 = 1.652 \times 10^3$$

Notice that the final zero in the decimal portion of 1.6520×10^3 is omitted.

13.6 Express the number in scientific notation: 4,390,000.

It is helpful to explicitly place the decimal after the ones digit before you convert the number into scientific notation.

$$4{,}390{,}000 = 4{,}390{,}000.0$$

Move the decimal point to the left, counting the number of digits you pass along the way, until only the leftmost digit 4 appears left of the decimal point. You will first pass the three digits 000, followed by the three digits 390.

$$4.3900000$$

The zeroes appearing at the end of the decimal, after the decimal point, should be omitted.

$$4.39$$

You moved the decimal point a total of six places to the left, so multiply 4.39 by 10 raised to the sixth power.

$$4{,}390{,}000 = 4.39 \times 10^6$$

13.7 Explain why the number below is not correctly expressed in scientific notation and correct the error.

$$13.74 \times 10^6$$

Only one non-zero digit may appear left of the decimal point in scientific notation, but this number has two digits (1 and 3) left of the decimal point. To correct the error, move the decimal point one place to the left, transforming 13.74 into 1.374.

You have to reflect the movement of the decimal point in the exponential power. As Problem 13.5 explains, each time a place value is passed by the decimal point as it moves left, you add one to the power of 10; thus, 10 should now be raised to the seventh power instead of the sixth power.

$$13.74 \times 10^6 = 1.374 \times 10^7$$

13.8 Express the number in standard notation: 6.57×10^{11}.

Consider Problems 13.5–13.6, which require you to write a number in scientific notation. In both examples, your final result is a decimal multiplied by 10 raised to a positive power. Furthermore, in each case, the power is equivalent to the number of digits you pass as you move the decimal point to the left.

To reconstitute the original number given scientific notation, work backwards; move the decimal point to the *right* a total of 11 places. Because 6.57 only contains two decimal digits, you will have to place nine additional zeroes at the end of the number; this allows you to move the decimal point a full 2 + 9 = 11 places to the right.

$$6.57 \times 10^{11} = 657000000000.0$$

Use commas to form groups of three digits, beginning at the right side of the number.

$$6.57 = 657{,}000{,}000{,}000$$

Thus, 6.57×10^{11} is equal to six hundred fifty-seven billion.

13.9 Express the number in scientific notation: 0.00081.

In order to express this number in scientific notation, exactly one non-zero digit must appear left of the decimal point. Unlike the preceding problems, this requires you to move the decimal point to the *right*. Count the digits passed by the decimal point as you move it to the right, as illustrated below.

Number	***Total digits passed by the decimal point***
0.00081	0
00.0081	1
000.081	2
0000.81	3
00008.1	4

In other words, use the decimal 8.1 instead of 00008.1.

Exactly one non-zero digit (8) now appears left of the decimal point. Omit the unnecessary zeroes when writing the number in scientific notation. You moved the decimal point four places to the right, so raise 10 to the –4 power. Note that the exponent is negative; this indicates the direction in which the decimal point was moved. In preceding problems, you moved the decimal point to the left, which corresponds with a positive exponent. Conversely, when you move the decimal point to the right, it results in a negative exponent.

$$0.00081 = 8.1 \times 10^{-4}$$

13.10 Express the number in scientific notation: 0.00000000009836.

Move the decimal point to the right until it passes exactly one non-zero digit. This will require you to pass ten 0 digits and a single 9 digit. In total, you must pass 11 digits to produce the decimal 9.836, which has exactly one non-zero digit left of the decimal point. Multiply that decimal by 10 raised to the –11 power to express the number in scientific notation. Remember that the exponent must be negative, because you moved the decimal point to the right.

$$0.00000000009836 = 9.836 \times 10^{-11}$$

13.11 Express the number in standard notation: 2.55×10^{-16}.

Consider Problems 13.9–13.10, in which you express values less than one in scientific notation. In each case, you move the decimal to the right in order to secure a single non-zero digit left of the decimal point. Because you move the decimal point to the right, 10 is raised to a negative power.

Therefore, to ensure the digit 2 appears left of the decimal point in this number, the decimal point had to be moved 16 places to the right. To express this number in standard notation, you have to move the decimal point back to its original position, 16 places to the left. Begin by moving the decimal point one place to the left.

$$0.255$$

To continue moving the decimal point the remaining 15 places, you will have to insert place holder zeroes.

$$0.000000000000000255$$

Look back at Problems 13.3 and 13.4: A positive exponent means a big number ($10^5 = 100{,}000$) and a negative exponent means a small number ($10^{-8} = 0.00000001$).

To the right of the decimal point, this decimal consists of fifteen zeroes followed by the digits 255. When you convert a number in scientific notation to standard notation and that number has a negative exponent, the result will be a decimal point followed by a series of zeroes that number one fewer than the exponent of 10—in this case, there are fifteen zeroes preceding the non-zero digits and 10 is raised to the sixteenth power.

Operations in Scientific Notation

+, −, ·, and ÷ numbers in scientific notation

13.12 Simplify the expression below using scientific notation, and verify your answer using standard notation.

$$(5.2 \times 10^2) + (1.4 \times 10^2)$$

This problem directs you to calculate the sum in two ways, first by leaving the numbers in scientific notation, and second by converting the numbers into standard notation to verify your answer.

Note that you cannot add numbers that are written in scientific notation unless 10 is raised to the same power in each. In this problem, 5.2×10^2 and 1.4×10^2 both contain 10 to the second power, so the requirement is met. Apply the following process to add numbers expressed in scientific notation.

1. Add the decimal portions of the numbers.
2. Multiply the sum by 10 raised to the shared power.
3. Ensure that exactly one non-zero digit appears left of the decimal point, and adjust if necessary.

Begin with Step 1, adding the decimal portions of the given numbers.

$$5.2 + 1.4 = 6.6$$

Multiply that sum by 10^2, the power of 10 shared by both of the original numbers.

$$6.6 \times 10^2$$

Because exactly one non-zero digit appears left of the decimal point, the sum is complete.

$$(5.2 \times 10^2) + (1.4 \times 10^2) = 6.6 \times 10^2$$

To verify that your answer is correct, express both of the numbers in standard notation: $5.2 \times 10^2 = 520$ and $1.4 \times 10^2 = 140$. Calculate the sum of the numbers.

Move the decimals two places to the right. When the power of 10 is positive, standard notation will be larger than the original decimal: 520 > 5.2 and 140 > 1.4. The opposite is true for negative exponents.

$$520 + 140 = 660$$

Finally, express 660.0 in scientific notation by moving the decimal point two places to the left.

$$660.0 = 6.6 \times 10^2$$

The sums, calculated in scientific and standard notation, are equal, so the solution is verified.

13.13 Simplify the expression: $(9.3 \times 10^{21}) - (4.7 \times 10^{21})$.

Like addition, subtraction of numbers in scientific notation is possible only when they share the same power of 10. In this problem, both 10s are raised to the 21 power. To calculate the difference, apply a procedure that mirrors the steps outlined in Problem 13.12.

1. Subtract the decimal portions of the numbers.
2. Multiply the sum by 10 raised to the shared power.
3. Ensure that exactly one non-zero digit appears left of the decimal point, and adjust if necessary.

Begin by subtracting the decimal portions of the numbers.

$$9.3 - 4.7 = 4.6$$

Multiply the difference by 10 raised to the shared exponent.

$$4.6 \times 10^{21}$$

Exactly one non-zero digit appears left of the decimal point, so the problem is complete.

$$(9.3 \times 10^{21}) - (4.7 \times 10^{21}) = 4.6 \times 10^{21}$$

13.14 Simplify the expression: $(8.43 \times 10^{13}) + (6.952 \times 10^{13})$.

Both of the numbers include 10 raised to the same power (13), so you can apply the steps described in Problem 13.12. Begin by adding the decimal portions of the numbers. Make sure to align the decimal points and place values, and insert a zero place holder into the thousandths place of 8.43 so that both numbers have the same number of decimal places, as illustrated below.

$$\begin{array}{r} 8.430 \\ +\,6.952 \\ \hline 15.382 \end{array}$$

Multiply the sum by 10 raised to the shared power, 13.

$$15.382 \times 10^{13}$$

Notice that the number is not correctly expressed in scientific notation, because two non-zero digits appear left of the decimal point. Apply the procedure described in Problem 13.7: Move the decimal point one additional place to the left, and reflect this movement by adding one to the exponent.

$$1.5382 \times 10^{14}$$

You conclude that $(8.43 \times 10^{13}) + (6.952 \times 10^{13}) = 1.5382 \times 10^{14}$.

13.15 Simplify the expression: $(52.75 \times 10^{7}) - (4.3 \times 10^{8})$.

Recall that numbers in scientific notation may only be added or subtracted if the powers of 10 are equal. In this problem, the numbers are raised to different powers, but notice that the first number, 52.75×10^{7}, is not written in scientific notation—two digits appear left of the decimal point. If you move the decimal point one place to the left, you must also add one to the exponent.

$$52.75 \times 10^{7} = 5.275 \times 10^{8}$$

The powers of 10 in both numbers are now equal.

$$(5.275 \times 10^{8}) - (4.3 \times 10^{8})$$

Subtract the decimal portions.

$$5.275 - 4.3 = 0.975$$

Multiply by the shared power of 10.

$$0.975 \times 10^{8}$$

The number is not written in scientific notation, because the only digit left of the decimal point is 0. Move the decimal point one place to the right, and subtract one from the power of 10 to reflect this action.

$$0.975 \times 10^8 = 9.75 \times 10^7$$

You conclude that $(52.75 \times 10^7) - (4.3 \times 10^8) = 9.75 \times 10^7$.

There's no symbol between the numbers, so you're supposed to multiply them.

13.16 Simplify the expression: $(7.1 \times 10^3)(2.8 \times 10^3)$.

Unlike addition and subtraction, numbers in scientific notation need not be raised to the same power of 10 in order to multiply or divide. Follow the steps below to calculate the product.

1. Multiply the decimal portions of the numbers.
2. Multiply the sum by 10 raised to the sum of the exponents.
3. Ensure that exactly one nonzero digit appears left of the decimal point, and adjust if necessary.

Add up the powers of 10. When you divide numbers in scientific notation, you'll subtract the numbers.

Begin by multiplying the decimal portions of the numbers.

$$(7.1)(2.8) = 19.88$$

Multiply the product by 10 raised to the sum of the exponents. In this problem, both 10s are raised to the third power.

$$19.88 \times 10^{3+3} = 19.88 \times 10^6$$

Move the decimal point one place to the left to express the number in scientific notation. Add one to the exponent of 10 to reflect the movement of the decimal point.

$$19.88 \times 10^6 = 1.988 \times 10^7$$

You conclude that $(7.1 \times 10^3)(2.8 \times 10^3) = 1.988 \times 10^7$.

13.17 Verify your solution to Problem 13.16 by calculating the product of the numbers written in standard notation.

To express the factors as numbers in standard notation, move the decimal point to the right three places in each, as directed by the exponent of 10.

$$7.1 \times 10^3 = 7{,}100$$
$$2.8 \times 10^3 = 2{,}800$$

Calculate the product of 7,100 and 2,800.

$$7{,}100 \times 2{,}800 = 19{,}880{,}000$$

Express the number in scientific notation by moving the decimal point of 19,880,000.0 seven places to the left.

$$19{,}880{,}000.0 = 1.988 \times 10^7$$

This answer matches the final result in Problem 13.16, so the solution is verified.

13.18 Simplify the expression: $(4.9 \times 10^{15})(6.302 \times 10^{-8})$.

Apply the procedure outlined in Problem 13.16, the first step of which is to multiply the decimal portions of the numbers.

$$(6.302)(4.9) = 30.8798$$

Multiply the product by 10 raised to the sum of the powers in the original expression: 15 and –8.

Adding 15 and -8 is the same as subtracting 8 from 15.

$$30.8798 \times 10^{15+(-8)} = 30.8798 \times 10^{7}$$

Express the number in scientific notation by moving the decimal point one place to the left.

$$30.8798 \times 10^{7} = 3.08798 \times 10^{8}$$

13.19 Simplify the expression: $(6.441 \times 10^{-7})(1.3 \times 10^{-9})$.

Multiply the decimal portions of the numbers.

$$(6.441)(1.3) = 8.3733$$

Multiply the product by 10 raised to the sum of the powers in the original expression: –7 and –9. Recall that the sum of two negative numbers is the sum of their absolute values multiplied by –1. Thus, $(-7) + (-9) = -16$.

If two numbers have the same sign (either both + or both -), ignore the signs, add the numbers, and then stick the matching sign in front of the sum.

$$8.3733 \times 10^{(-7)+(-9)} = 8.3733 \times 10^{-16}$$

13.20 Simplify the expression: $(15.68 \times 10^{7}) \div (9.8 \times 10^{5})$.

Follow the steps below to divide two numbers expressed in scientific notation.

1. Divide the decimal portions of the numbers.
2. Multiply the sum by 10 raised to the difference of the exponents.
3. Ensure that exactly one non-zero digit appears left of the decimal point, and adjust if necessary.

Begin by calculating the quotient of the decimal portions of the numbers using long division. As numerous examples of long division have been demonstrated throughout the preceding chapters, this step is omitted.

Not to be insensitive, but if you can't long divide by yourself at this point, you really need to go back and review it. It needs to be automatic by now.

$$15.68 \div 9.8 = 1.6$$

Unlike the process used to multiply numbers in scientific notation (described in Problem 13.16), the power of 10 is equal to the *difference* of the exponents 7 and 5, not the sum. Ensure that you subtract the power of the divisor (5) from the power of the dividend (7), regardless of the values of these numbers. For instance, if the power of the divisor is greater than the power of the dividend, then the difference will be a negative number. Multiply the quotient calculated above by the difference of the exponents.

$$1.6 \times 10^{7-5} = 1.6 \times 10^2$$

You conclude that $(15.68 \times 10^7) \div (9.8 \times 10^5) = 1.6 \times 10^2$. Because $1.6 \times 10^2 = 160$, that answer is also correct.

13.21 Simplify the expression: $(1.118 \times 10^{13}) \div (2.6 \times 10^{-4})$.

Calculate the quotient of the decimal portions of the numbers.

$$1.118 \div 2.6 = 0.43$$

Multiply the quotient by the difference of the exponents. Note that you are subtracting a negative number from a positive number: $13 - (-4)$. According to Problem 3.23, two adjacent negative signs should be replaced with a single positive sign: $13 - (-4) = 13 + 4$.

$$\begin{aligned} 0.43 \times 10^{13-(-4)} &= 0.43 \times 10^{13+4} \\ &= 0.43 \times 10^{17} \end{aligned}$$

Note that the number is not in scientific notation, because the only digit left of the decimal point is zero. Move the decimal point one place to the right and subtract one from the exponent of 10.

$$0.43 \times 10^{17} = 4.3 \times 10^{16}$$

Therefore, $(1.118 \times 10^{13}) \div (2.6 \times 10^{-4}) = 4.3 \times 10^{16}$.

13.22 Simplify the expression: $(3.1694 \times 10^{-15}) \div (2.99 \times 10^{-20})$.

Divide the decimal portions of the numbers.

$$3.1694 \div 2.99 = 1.06$$

Multiply the quotient by 10 raised to the difference of the powers. Note that $-15 - (-20) = -15 + 20$, as explained in Problem 13.21.

$$\begin{aligned} 1.06 \times 10^{-15-(-20)} &= 1.06 \times 10^{-15+20} \\ &= 1.06 \times 10^5 \end{aligned}$$

You conclude that $(3.1694 \times 10^{-15}) \div (2.99 \times 10^{-20}) = 1.06 \times 10^5$.

Whole Number Exponents

Tricks and techniques to simplify exponential expressions

13.23 Calculate x in the expression below and describe the rule that allows you to calculate x without expanding the exponential expressions.

$$2^3 \cdot 2^5 = 2^x$$

When two exponential expressions with the same base are multiplied, the product is equal to the shared base raised to the sum of the powers. In this problem, on the left side of the equation, a common base of 2 is raised to two different powers: 3 and 5. To calculate the product of 2^3 and 2^5, raise the common base of 2 to the sum of the powers.

$$\begin{aligned} 2^3 \cdot 2^5 &= 2^{3+5} \\ &= 2^8 \end{aligned}$$

Thus, $2^3 \cdot 2^5 = 2^8$. Notice that the 8 in this expression and the x in the original expression appear in the same, corresponding location, and that the expressions are otherwise equal. Therefore, you conclude that $x = 8$.

You can only add exponents to simplify an expression, as described above, if two conditions are met:

- Both exponential expressions have the same base. In other words, the numbers that are raised to exponents in both expressions are equal.
- The exponential expressions must be multiplied. For example, $3^2 \cdot 3^8 = 3^{10}$ because you are calculating a product, but $3^2 + 3^8 \neq 3^{10}$.

You will revisit this technique throughout the remainder of the book, especially upon the introduction of variables in Chapter 15.

In other words, figure out what 2^3 and 2^5 are equal to and then multiply those numbers.

13.24 Verify your answer to Problem 13.23 by expanding the exponential expressions and calculating the product.

According to Problem 13.23, $2^3 \cdot 2^5 = 2^8$. Expand the exponential expressions 2^3, 2^5, and 2^8, multiplying two factors at a time and working left to right.

$$\begin{aligned} 2^3 &= 2\cdot 2\cdot 2 \\ &= 4\cdot 2 \\ &= 8 \end{aligned} \qquad \begin{aligned} 2^5 &= 2\cdot 2\cdot 2\cdot 2\cdot 2 \\ &= 4\cdot 2\cdot 2\cdot 2 \\ &= 8\cdot 2\cdot 2 \\ &= 16\cdot 2 \\ &= 32 \end{aligned} \qquad \begin{aligned} 2^8 &= 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2 \\ &= 4\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2 \\ &= 8\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2 \\ &= 16\cdot 2\cdot 2\cdot 2\cdot 2 \\ &= 32\cdot 2\cdot 2\cdot 2 \\ &= 64\cdot 2\cdot 2 \\ &= 128\cdot 2 \\ &= 256 \end{aligned}$$

Substitute the values calculated above into the expression $2^3 \cdot 2^5 = 2^8$ to verify that the equation remains true.

$$\begin{aligned} 2^3 \cdot 2^5 &= 2^8 \\ 8 \cdot 32 &= 256 \\ 256 &= 256 \end{aligned}$$

Because both sides of the equation are equal, you conclude that the solution to Problem 13.23 is correct.

13.25 Calculate x in the expression below and describe the rule that allows you to calculate x without expanding the exponential expressions.

$$\frac{9^{13}}{9^6} = 9^x$$

According to Problem 13.23, to multiply two exponential expressions with the same base, you raise the shared base to the sum of the exponential powers. To *divide* two exponential expressions with the same base, raise the shared base to the *difference* of the exponential powers. To calculate the quotient of 9^{13} and 9^6, subtract the exponent of the divisor from the exponent of the dividend.

$$\begin{aligned} \frac{9^{13}}{9^6} &= 9^{13} \div 9^6 \\ &= 9^{13-6} \\ &= 9^7 \end{aligned}$$

Because $\frac{9^{13}}{9^6} = 9^x = 9^7$, you conclude that $x = 7$.

13.26 Determine whether the following statement is true or false and explain your answer.

$$\left(3^4\right)^7 = 3^{28}$$

To verify this statement, recall that raising 3^4 to the power of 7 is the equivalent of multiplying seven of the quantities 3^4 together, as illustrated below.

$$\left(3^4\right)^7 = 3^4 \cdot 3^4 \cdot 3^4 \cdot 3^4 \cdot 3^4 \cdot 3^4 \cdot 3^4$$

According to Problem 13.23, the product of exponential expressions with the same base is the shared base raised to the sum of the powers.

$$\begin{aligned} 3^4 \cdot 3^4 \cdot 3^4 \cdot 3^4 \cdot 3^4 \cdot 3^4 \cdot 3^4 &= 3^{4+4+4+4+4+4+4} \\ &= 3^{28} \end{aligned}$$

Therefore, the statement $\left(3^4\right)^7 = 3^{28}$ is true. This property of exponents can be summarized as follows: If a number raised to an exponent is, in turn, raised to an exponent, the expression is equal to the base raised to the product of

Think of it this way: When a power is raised to a power, multiply the powers together.

the exponents. In this Problem, 3 is raised to the 4 power, and that entire expression is raised to the 7 power. The answer is 3 raised to the $4 \cdot 7 = 28$ power.

$$\begin{aligned}\left(3^4\right)^7 &= 3^{4 \cdot 7}\\ &= 3^{28}\end{aligned}$$

In other words, rewrite 32^{11} as 2 raised to some power. That power will be x.

13.27 Identify the value of x that correctly completes the statement: $32^{11} = 2^x$.

Your goal in this problem is to express 32^{11} as an exponential expression with base 2. It is helpful to list powers of 2.

$$\begin{aligned}2^1 &= 2\\ 2^2 &= 2 \cdot 2 = 4\\ 2^3 &= 2 \cdot 2 \cdot 2 = 8\\ 2^4 &= 2 \cdot 2 \cdot 2 \cdot 2 = 16\\ 2^5 &= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32\end{aligned}$$

Replace 32 in the original equation with the equivalent expression 2^5.

$$\begin{aligned}32^{11} &= 2^x\\ \left(2^5\right)^{11} &= 2^x\end{aligned}$$

Multiply the exponents on the left side of the equation to identify the value of x.

$$\begin{aligned}2^{5 \cdot 11} &= 2^x\\ 2^{55} &= 2^x\end{aligned}$$

You conclude that $x = 55$.

13.28 Apply the exponential rules described in Problems 13.25–13.26 to simplify the statement below without expanding the exponential expressions.

$$10^{15} \div 100^3$$

The rules described in this section apply only to exponential expressions that share a common base. Thus, you must first express 10^{15} and 100^3 in terms of the same base. Because $100 = 10^2$, you can substitute the expression 10^2 for 100 in the original expression.

$$10^{15} \div 100^3 = 10^{15} \div \left(10^2\right)^3$$

According to Problem 13.26, $\left(10^2\right)^3 = 10^{2 \cdot 3} = 10^6$.

$$10^{15} \div \left(10^2\right)^3 = 10^{15} \div 10^6$$

Problem 13.25 states that the quotient of exponential expressions with the same base is equal to the shared base raised to the difference of the powers.

$$\begin{aligned}10^{15} \div 10^{6} &= 10^{15-6}\\ &= 10^{9}\end{aligned}$$

You conclude that $10^{15} \div 100^{3} = 10^{9}$.

13.29 Calculate x in the following statement: $\dfrac{36^5}{216^8} = 6^x$.

The problem directs you to rewrite the expression as a single exponential expression with base 6. Begin by listing consecutive powers of 6.

$$\begin{aligned}6^1 &= 6\\ 6^2 &= 6\cdot 6 = 36\\ 6^3 &= 6\cdot 6\cdot 6 = 216\end{aligned}$$

Replace 36 in the original equation with the equivalent value 6^2; similarly, replace 216 with 6^3.

$$\frac{36^5}{216^8} = 6^x$$

$$\frac{\left(6^2\right)^5}{\left(6^3\right)^8} = 6^x$$

Exponential expressions raised to powers are equal to the bases raised to the products of the powers.

$$\frac{6^{2\cdot 5}}{6^{3\cdot 8}} = 6^x$$

$$\frac{6^{10}}{6^{24}} = 6^x$$

The quotient of two exponential expressions with the same base is equal to the shared base raised to the difference of the powers.

$$6^{10-24} = 6^x$$

$$6^{-14} = 6^x$$

Problems 13.32–13.40 deal with negative exponents; that's when you'll find out what 6^{-14} actually represents.

You conclude that $x = -14$.

13.30 Calculate x in the following statement: $256^9 \div 16^3 \cdot 64^6 = 4^x$.

Because you are directed to express a product as a single power of 4, list consecutive powers of 4.

$$\begin{aligned}4^1 &= 4\\ 4^2 &= 4\cdot 4 = 16\\ 4^3 &= 4\cdot 4\cdot 4 = 64\\ 4^4 &= 4\cdot 4\cdot 4\cdot 4 = 256\end{aligned}$$

Replace each base on the left side of the original equation with the equivalent exponential expression and simplify the expression.

$$\begin{aligned} 256^9 \div 16^3 \cdot 64^6 &= 4^x \\ \left(4^4\right)^9 \div \left(4^2\right)^3 \cdot \left(4^3\right)^6 &= 4^x \\ 4^{4\cdot 9} \div 4^{2\cdot 3} \cdot 4^{3\cdot 6} &= 4^x \\ 4^{36} \div 4^6 \cdot 4^{18} &= 4^x \end{aligned}$$

Work from left to right, first calculating the quotient $4^{36} \div 4^6$, and then multiplying the result by 4^{18}.

$$\begin{aligned} 4^{36-6} \cdot 4^{18} &= 4^x \\ 4^{30} \cdot 4^{18} &= 4^x \\ 4^{30+18} &= 4^x \\ 4^{48} &= 4^x \end{aligned}$$

Therefore, $x = 48$.

13.31 Calculate x in the following statement: $\left(\frac{1{,}024^3}{64^{12}} \cdot 128^8\right)^2 = 2^x$.

Each of the bases in the expression is a power of two: $1{,}024 = 2^{10}$, $64 = 2^6$, and $128 = 2^7$. Substitute the equivalent values into the equation.

$$\begin{aligned} \left(\frac{1{,}024^3}{64^{12}} \cdot 128^8\right)^2 &= 2^x \\ \left(\frac{\left[2^{10}\right]^3}{\left[2^6\right]^{12}} \cdot \left[2^7\right]^8\right)^2 &= 2^x \\ \left(\frac{2^{10\cdot 3}}{2^{6\cdot 12}} \cdot 2^{7\cdot 8}\right)^2 &= 2^x \\ \left(\frac{2^{30}}{2^{72}} \cdot 2^{56}\right)^2 &= 2^x \end{aligned}$$

Subtract exponents of common bases to calculate quotients; add exponents to calculate products.

$$\begin{aligned} \left(2^{30-72} \cdot 2^{56}\right)^2 &= 2^x \\ \left(2^{-42} \cdot 2^{56}\right)^2 &= 2^x \\ \left(2^{-42+56}\right)^2 &= 2^x \\ \left(2^{14}\right)^2 &= 2^x \\ 2^{28} &= 2^x \end{aligned}$$

You conclude that $x = 28$.

Negative Exponents

The moving case for negative powers

13.32 Identify the equivalent fraction: 3^{-1}.

Any integer can be expressed as a fraction by dividing the integer by 1. Thus, $3=\frac{3}{1}$. Replace 3 in the given expression with the equivalent fraction.

$$3^{-1}=\left(\frac{3}{1}\right)^{-1}$$

Raising a fraction to the –1 power is the equivalent of taking its reciprocal. Recall that the reciprocal of a fraction reverses the numerator and denominator.

$$\left(\frac{3}{1}\right)^{-1}=\frac{1}{3}$$

You conclude that $3^{-1}=\frac{1}{3}$.

If an entire fraction is raised to a negative power, flip the fraction upside down. If an integer (like 3 in this problem) is raised to a negative power, make it into a fraction (stick on a denominator of 1) and flip it upside down.

13.33 Demonstrate that $2^{-1}=\frac{1}{2}$ by calculating the quotients $2 \div 4$ and $2 \div 2^2$.

Write the quotient $2 \div 4$ as a fraction reduced to lowest terms.

$$\begin{aligned}2\div 4&=\frac{2}{4}\\&=\frac{2\div 2}{4\div 2}\\&=\frac{1}{2}\end{aligned}$$

The problem asks you to calculate the same quotient a second time, using the techniques outlined in Problems 13.23–13.31. This expression writes the divisor 4 as a power of 2 so both exponential expressions have a common base. This allows you to subtract the powers of the quotient.

$$\begin{aligned}2\div 2^2&=2^1\div 2^2\\&=2^{1-2}\\&=2^{-1}\end{aligned}$$

In the steps above, you have determined that $2\div 4=\frac{1}{2}$ and $2 \div 2^2 = 2^{-1}$. Because $\frac{1}{2}$ and 2^{-1} are both equal to the same expression $(2 \div 4)$, you can conclude that those values are also equal to each other.

$$\frac{1}{2}=2^{-1}$$

This result verifies the solution to Problem 13.32, which states that $3^{-1}=\frac{1}{3}$. Raising an integer to the –1 power results in the reciprocal.

10^{-1} is equal to 1 divided by 10^1. That means 10^{-2} is equal to 1 divided by 10^2. Basically, a negative exponent means a number has to be moved across the fraction bar, from the numerator to the denominator or vice versa. Once you move the number to the correct part of the fraction, the exponent is once again positive.

13.34 Simplify the expressions below: (a) 10^{-1} and (b) 10^{-2}.

As explained in Problems 13.32–13.33, raising an integer to the –1 power results in the reciprocal.

$$\begin{aligned}10^{-1} &= \left(\frac{10}{1}\right)^{-1}\\ &= \frac{1}{10}\end{aligned}$$

Raising an integer to any negative power has an effect similar to raising the integer to the –1 power. Like 10^{-1}, the fraction equivalent of 10^{-2} also has a numerator of 1. However, the power of the 10 in its denominator is 2, rather than 1.

$$\begin{aligned}10^{-2} &= \frac{1}{10^2}\\ &= \frac{1}{100}\end{aligned}$$

You conclude that $10^{-2} = \frac{1}{100}$.

13.35 Simplify the expression: $\left(\frac{4}{5}\right)^{-1}$.

A simplified expression does not contain any negative exponents.

As first explained in Problem 13.32, raising a fraction to a negative power is equivalent to calculating the reciprocal of the fraction. Once the fraction is rewritten as its reciprocal, take the absolute value of the exponent, eliminating the negative sign. Thus, once $\frac{4}{5}$ is rewritten as $\frac{5}{4}$, the –1 exponent is changed to +1.

$$\left(\frac{4}{5}\right)^{-1} = \left(\frac{5}{4}\right)^{1}$$

Raising any number to the first power does not affect its value: $\left(\frac{5}{4}\right)^{1} = \frac{5}{4}$. You conclude that $\left(\frac{4}{5}\right)^{-1} = \frac{5}{4}$.

13.36 Simplify the expression: $\left(\frac{7}{6}\right)^{-2}$.

The fraction is raised to a negative exponent, so replace the fraction with its reciprocal and take the absolute value of the exponent.

$$\left(\frac{7}{6}\right)^{-2} = \left(\frac{6}{7}\right)^{2}$$

The entire fraction is raised to the second power. To simplify the expression, the numerator and denominator should each be raised to the second power independently.

$$\left(\frac{6}{7}\right)^2=\frac{6^2}{7^2}$$
$$=\frac{6\cdot 6}{7\cdot 7}$$
$$=\frac{36}{49}$$

13.37 Simplify the expression: $5^3\cdot 9^{-2}$.

Two quantities are multiplied in this expression, but only one is raised to a negative power. Rewrite that quantity, 9^{-2}, as its reciprocal and simplify.

$$5^3\cdot 9^{-2}=5^3\cdot\left(\frac{9}{1}\right)^{-2}$$
$$=5^3\cdot\left(\frac{1}{9}\right)^2$$
$$=5^3\cdot\frac{1^2}{9^2}$$
$$=5^3\cdot\frac{1}{81}$$
$$=125\cdot\frac{1}{81}$$

To multiply an integer (like 125) and a fraction, first rewrite the integer as a fraction. The product of two fractions is equal to the product of the numerators divided by the product of the denominators.

$$=\frac{125}{1}\cdot\frac{1}{81}$$
$$=\frac{125\cdot 1}{1\cdot 81}$$
$$=\frac{125}{81}$$

13.38 Simplify the expression: $4^{-3}\cdot 2^7\cdot\left(\frac{8}{3}\right)^{-2}$.

Rewrite each exponential expression that has a negative power as its reciprocal and simplify.

$$4^{-3}\cdot 2^7\cdot\left(\frac{8}{3}\right)^{-2}=\frac{1}{4^3}\cdot 2^7\cdot\left(\frac{3}{8}\right)^2$$
$$=\frac{1}{64}\cdot\frac{128}{1}\cdot\frac{9}{64}$$

Reduce the product before you multiply the fractions, noting that 64 and 128 share greatest common factor 64.

$$= \frac{1}{1\,\cancel{64}} \cdot \frac{2\,\cancel{128}}{1} \cdot \frac{9}{64}$$
$$= \frac{1}{1} \cdot \frac{2}{1} \cdot \frac{9}{64}$$

Numerator value 2 and denominator value 64 share greatest common factor 2.

$$= \frac{1}{1} \cdot \frac{1\,\cancel{2}}{1} \cdot \frac{9}{32\,\cancel{64}}$$
$$= \frac{1}{1} \cdot \frac{1}{1} \cdot \frac{9}{32}$$
$$= \frac{9}{32}$$

You conclude that $4^{-3} \cdot 2^{7} \cdot \left(\frac{8}{3}\right)^{-2} = \frac{9}{32}$.

13.39 Simplify the expression: $\frac{3^3 \cdot 2^{-4}}{6^{-2}}$.

Exponential expressions with negative exponents should be moved across the fraction bar; this is the equivalent of applying the reciprocal. Thus, 2^{-4} should be moved to the denominator and the absolute value of its power taken, resulting in 2^4. Similarly, 6^{-2} should be moved to the numerator, and the absolute value of its power taken as well.

You can only "move" expressions with negative exponents across the fraction bar when you're multiplying. If the numerator were $3^3 + 2^{-4}$, you wouldn't be able to move 2^{-4} to the denominator and make its exponent positive.

$$\frac{3^3 \cdot 2^{-4}}{6^{-2}} = \frac{3^3 \cdot 6^2}{2^4}$$
$$= \frac{27 \cdot 36}{16}$$

Numerator value 36 and denominator value 16 share greatest common factor 4.

$$= \frac{27 \cdot \cancel{36}\,9}{\cancel{16}\,4}$$
$$= \frac{27 \cdot 9}{4}$$
$$= \frac{243}{4}$$

13.40 Simplify the expression: $4^{-3} \cdot 3^2 \cdot 2^{-1} \cdot 9^0$.

Any real number (other than zero), when raised to the power of zero, is equal to one. In this expression, $9^0 = 1$. Rewrite expressions with negative exponents as their reciprocals raised to positive powers.

$$\begin{aligned}4^{-3} \cdot 3^2 \cdot 2^{-1} \cdot 9^0 &= \frac{1}{4^3} \cdot 3^2 \cdot \frac{1}{2} \cdot 1\\ &= \frac{1}{64} \cdot \frac{9}{1} \cdot \frac{1}{2} \cdot \frac{1}{1}\\ &= \frac{1 \cdot 9 \cdot 1 \cdot 1}{64 \cdot 1 \cdot 2 \cdot 1}\\ &= \frac{9}{128}\end{aligned}$$

Chapter 14

ROOTS AND RADICALS

The opposites of exponents

Chapter 13 explores the concept of exponents—notation that represents repeated multiplication. This chapter explores the inverse of exponents, roots. Basic algebra focuses primarily on two types of roots: square roots and cube roots. However, you will also investigate other roots, and apply similar principles to simplify those expressions. The chapter concludes by explicitly tying roots to exponents by expressing the former as the latter using fractional exponents.

In Chapter 13, you review exponents. Actually, you do more than review exponents—you go into quite a bit of depth. This chapter goes back to the basics of exponents but puts everything in reverse. For example, you already know that $3^2 = 9$, but after you've read this chapter, you'll understand why the square root of 9 is 3.

You might already be familiar with square roots, but this chapter also discusses cube roots and other roots that don't have nicknames (like the fourth root and the fifth root). Don't worry if you haven't heard of those—the same rules apply. To wrap up Chapter 14, you'll translate fractional exponents into roots (which are also called radical expressions).

Square Roots

If $x \cdot x = y$, then $\sqrt{y} = x$

14.1 Describe the relationship between perfect squares and their square roots.

Some books call them square numbers.

Perfect squares are integers that are equal to the product of some integer and itself. For example, calculate the product of the integer 2 and itself to identify a perfect square: $2 \cdot 2 = 4$. Because 4 is equal to the product of an integer (2) and itself, 4 is a perfect square.

The square root of a perfect square is equal to the number that is multiplied twice in order to produce the perfect square. Square roots are represented using the radical symbol "$\sqrt{\ }$." To summarize this example mathematically, you conclude $\sqrt{4} = 2$.

Technically, 0 is the smallest perfect square, because $0 \cdot 0 = 0$.

14.2 List the twenty smallest perfect squares, beginning with 1.

Calculate the squares of all consecutive integers beginning with 1 and ending with 20.

$1^2 = 1$	$8^2 = 64$	$15^2 = 225$
$2^2 = 4$	$9^2 = 81$	$16^2 = 256$
$3^2 = 9$	$10^2 = 100$	$17^2 = 289$
$4^2 = 16$	$11^2 = 121$	$18^2 = 324$
$5^2 = 25$	$12^2 = 144$	$19^2 = 361$
$6^2 = 36$	$13^2 = 169$	$20^2 = 400$
$7^2 = 49$	$14^2 = 196$	

Memorizing these perfect squares significantly increases the ease with which you simplify square roots, so make a point of committing all of these perfect squares to memory, especially the squares of the integers between and including 0 and 15.

14.3 Calculate the square root: $\sqrt{64}$.

Consult the list of perfect squares in Problem 14.2. Notice that 64 is a perfect square, as $8^2 = 64$. The square root of a number is equal to the value which must be squared in order to produce that number. In this case, because $8^2 = 64$, you conclude that $\sqrt{64} = 8$.

14.4 Calculate the square root: $\sqrt{121}$.

Among the perfect squares listed in Problem 14.2, you find $11^2 = 121$. If the square of 11 is 121, then the square root of 121 is 11.

$$\sqrt{121} = 11$$

14.5 To simplify square roots whose radicands are not perfect squares, you must apply the following property of roots: The square root of a product is equal to the product of the individual square roots. Demonstrate the validity of this statement using the equation below.

$$\sqrt{9\cdot 25}=\sqrt{9}\cdot\sqrt{25}$$

The radicand is the number (or expression) inside the square root symbol. If the radicand isn't a perfect square, your answer will contain a radical symbol—it won't vanish like in Problems 14.3–14.4.

Simplify the left side of the equation by calculating the product within the radical. Note that the product of two perfect squares (9 and 25) is also a perfect square (225).

$$\sqrt{9\cdot 25}=\sqrt{9}\cdot\sqrt{25}$$
$$\sqrt{225}=\sqrt{9}\cdot\sqrt{25}$$

Calculate all three square roots in the equation, noting that $15^2 = 225$, $3^2 = 9$, and $5^2 = 25$.

$$15 = 3\cdot 5$$

Because the above statement is true, you conclude that the original statement, $\sqrt{9\cdot 25}=\sqrt{9}\cdot\sqrt{25}$, is true as well.

14.6 Simplify the expression: $\sqrt{18}$.

When a radicand is not a perfect square, your goal is to identify factors of the radicand that are perfect squares. Among the factors of 18 (1, 2, 3, 6, 9, and 18), the largest perfect square is 9. Express 18 as a product of two numbers, one of which is 9.

In other words, divide 18 by 9 to get 2 and then write 18 as $9\cdot 2$.

$$\sqrt{18}=\sqrt{9\cdot 2}$$

As explained in Problem 14.5, the square root of a product is equal to the product of the individual square roots.

$$\sqrt{9\cdot 2}=\sqrt{9}\cdot\sqrt{2}$$

Calculate the square root of 9.

$$\sqrt{9}\cdot\sqrt{2}=3\cdot\sqrt{2}$$

Mathematics convention omits the multiplication dot from the final product; you conclude that $\sqrt{18}=3\sqrt{2}$.

14.7 Simplify the expression: $\sqrt{28}$.

The factors of 28 are 1, 2, 4, 7, 14, and 28. The largest perfect square among those factors is 4, so write 28 as a product of two integers, one of which is 4.

$$\sqrt{28} = \sqrt{4 \cdot 7}$$

Express the root of the product as a product of the roots and simplify.

$$\begin{aligned} &= \sqrt{4} \cdot \sqrt{7} \\ &= 2 \cdot \sqrt{7} \\ &= 2\sqrt{7} \end{aligned}$$

Thus, $\sqrt{28} = 2\sqrt{7}$.

14.8 Simplify the expression: $\sqrt{72}$.

Consider the perfect squares listed in Problem 14.2. Whereas 9 is a factor of 72, it is not the largest perfect square factor. Write 72 as a product of two numbers, one of which is its largest perfect square factor: 36.

$$\begin{aligned} \sqrt{72} &= \sqrt{36 \cdot 2} \\ &= \sqrt{36} \cdot \sqrt{2} \\ &= 6\sqrt{2} \end{aligned}$$

They have to be the same kind of roots, though. For example, if you're multiplying a square root by a cube root, this process doesn't apply.

14.9 Simplify the expression: $\sqrt{2}\left(\sqrt{8}\right)$.

According to Problem 14.5, the square root of a product is equal to the product of the square roots. The reverse is also true: The product of two square roots is equal to the square root of the product of the radicands. In this problem, the square roots of 2 and 8 are multiplied, and that product is equivalent to the square root of $2 \cdot 8$.

$$\begin{aligned} \sqrt{2}\left(\sqrt{8}\right) &= \sqrt{2 \cdot 8} \\ &= \sqrt{16} \\ &= 4 \end{aligned}$$

You conclude that $\sqrt{2}\left(\sqrt{8}\right) = 4$.

14.10 Simplify the expression: $\sqrt{5}\left(\sqrt{90}\right)$.

Like Problem 14.9, you are given a product of square roots, which is equal to the square root of the product of the radicands 5 and 90. However, given large radicands such as 90, it is often helpful to simplify the square root before multiplying. Notice that the perfect square 9 is a factor of 90.

$$\begin{aligned}\sqrt{5}\left(\sqrt{90}\right) &= \sqrt{5}\left(\sqrt{9\cdot 10}\right)\\ &= \sqrt{5}\left(\sqrt{9}\cdot\sqrt{10}\right)\\ &= \sqrt{5}\left(3\sqrt{10}\right)\end{aligned}$$

The resulting expression still contains a product of square roots. To organize the product visually, you may choose to write those square roots next to one another.

$$= 3\sqrt{5}\sqrt{10}$$

The product of the square roots is equal to the square root of the product of the radicands.

$$\begin{aligned}&= 3\sqrt{5\cdot 10}\\ &= 3\sqrt{50}\end{aligned}$$

Simplify the square root.

$$\begin{aligned}&= 3\sqrt{25\cdot 2}\\ &= 3\sqrt{25}\sqrt{2}\\ &= 3\cdot 5\sqrt{2}\end{aligned}$$

Multiply the integers to complete the problem.

$$= 15\sqrt{2}$$

14.11 Simplify the expression: $\sqrt{7}\left(\sqrt{32}\right)$.

The square root within parentheses can be simplified, because one of its factors (16) is a perfect square.

$$\begin{aligned}\sqrt{7}\left(\sqrt{32}\right) &= \sqrt{7}\left(\sqrt{16\cdot 2}\right)\\ &= \sqrt{7}\left(\sqrt{16}\cdot\sqrt{2}\right)\\ &= \sqrt{7}\left(4\sqrt{2}\right)\\ &= 4\sqrt{7}\sqrt{2}\end{aligned}$$

The product of two square roots is equal to the square root of the product of the radicands.

$$\begin{aligned}&= 4\sqrt{7\cdot 2}\\ &= 4\sqrt{14}\end{aligned}$$

The radical expression $\sqrt{14}$ cannot be simplified, because 14 has no factors (other than 1, which is a factor of all integers) that are perfect squares.

14.12 Simplify the expression: $\frac{\sqrt{24}}{\sqrt{6}}$.

Instead of two separate radicals, draw one big radical symbol that covers the entire fraction.

According to Problem 14.9, the product of two square roots is equal to the square root of the product of the radicands. Similarly, the quotient of two square roots is equal to the square root of the quotient of the radicands.

$$\frac{\sqrt{24}}{\sqrt{6}} = \sqrt{\frac{24}{6}}$$

Reduce the fraction to lowest terms and simplify.

$$\begin{aligned} &= \sqrt{\frac{24 \div 6}{6 \div 6}} \\ &= \sqrt{\frac{4}{1}} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

14.13 Simplify the expression: $3\sqrt{11} + 7\sqrt{11}$.

While you are able to multiply and divide square roots simply by multiplying or dividing their radicands and then taking the square root, as demonstrated in Problems 14.9 and 14.12, the same is not true for addition. In order to add square roots, the radicands of the square roots must be equal. In this problem, both of the square roots have radicand 11, so that condition is satisfied.

The other significant difference between adding (or subtracting) square roots and the operations of multiplication and division is the treatment of the radicands. Whereas you multiply radicands when multiplying square roots, you *do not* add the radicands when adding square roots. Rather, you add the coefficients—the numbers multiplied by and immediately preceding—of the square roots.

Thus, to simplify this expression, add the coefficients 3 and 7; the result is multiplied by the common radical expression $\sqrt{11}$, as demonstrated below.

$$\begin{aligned} 3\sqrt{11} + 7\sqrt{11} &= (3+7)\sqrt{11} \\ &= 10\sqrt{11} \end{aligned}$$

14.14 Simplify the expression: $\sqrt{12} + \sqrt{27}$.

According to Problem 14.13, you cannot add square roots unless they share the same radicand. The radicands in this expression (12 and 27) are not equal, so at first glance, it appears as though the expression cannot be simplified. However, each of the radicands has a perfect square factor, allowing you to simplify.

$$\begin{aligned}\sqrt{12}+\sqrt{27}&=\sqrt{4\cdot 3}+\sqrt{9\cdot 3}\\&=\left(\sqrt{4}\cdot\sqrt{3}\right)+\left(\sqrt{9}\cdot\sqrt{3}\right)\\&=2\sqrt{3}+3\sqrt{3}\end{aligned}$$

The radicands of the sum are now equal. Add the coefficients of the radical expressions and multiply by the shared radical.

$$\begin{aligned}&=(2+3)\sqrt{3}\\&=5\sqrt{3}\end{aligned}$$

Cube Roots

If $a\cdot a\cdot a=b$, then $\sqrt[3]{b}=a$

Perfect squares are always positive, but perfect cubes can be negative. If you multiply a negative number times itself three times, the final answer is negative.

14.15 List the ten smallest positive perfect cubes, beginning with 1.

Memorizing the ten smallest positive perfect cubes is worthwhile, because the speed at which you recognize perfect cube factors of radicands is directly proportional to the speed at which you can simplify cube roots.

$1^3=1$	$6^3=216$
$2^3=8$	$7^3=343$
$3^3=27$	$8^3=512$
$4^3=64$	$9^3=729$
$5^3=125$	$10^3=1{,}000$

For each positive cube root there exists a corresponding negative cube root. For example, $4^3=4\cdot 4\cdot 4=64$ and $(-4)^3=(-4)(-4)(-4)=-64$.

A cube root looks like a square root, but it has a tiny little 3 called the "index" tucked into the "checkmark" part of the radical symbol. Square roots don't need an index number, but all other roots do.

14.16 Calculate the cube root: $\sqrt[3]{125}$.

Consider the list of perfect cubes in Problem 14.15. Each perfect cube is associated with the number that must be cubed in order to produce that value; for instance, $5^3=125$. Because cubing 5 produces 125, the cube root of 125 is 5: $\sqrt[3]{125}=5$.

14.17 Calculate the cube root: $\sqrt[3]{-729}$.

According to Problem 14.15, $9^3=729$. Therefore, 729 is a perfect cube and the cube root of 729 is 9: $\sqrt[3]{729}=9$. Furthermore, the cube root of the opposite of 729 is equal to the opposite of 9: $\sqrt[3]{-729}=-9$.

14.18 Simplify the expression: $\sqrt[3]{54}$.

This is the same thing you do in Problem 14.6, but now you're looking for perfect cube factors instead of perfect square factors.

Unlike Problems 14.16–14.17, the radicand of this expression (54) is not a perfect cube. Therefore, to simplify the radical, your goal is to identify factors of 54 that are perfect cubes. List the factors of 54.

$$1, 2, 3, 6, 9, 18, 27, 54$$

Of those factors, the largest perfect cube is 27. Write 54 as a product of two numbers, one of which is 27.

$$\sqrt[3]{54} = \sqrt[3]{27 \cdot 2}$$

The cube root of a product is equal to the product of the individual cube roots.

$$\begin{aligned}\sqrt[3]{27 \cdot 2} &= \sqrt[3]{27} \cdot \sqrt[3]{2} \\ &= 3 \cdot \sqrt[3]{2} \\ &= 3\sqrt[3]{2}\end{aligned}$$

14.19 Simplify the expression: $\sqrt[3]{40}$.

To simplify the cube root, identify perfect cube factors of the radicand. The largest perfect cube factor of 40 is 8, so rewrite 40 as the product of 8 and another integer by dividing 40 by 8: $40 \div 8 = 5$.

$$\begin{aligned}\sqrt[3]{40} &= \sqrt[3]{8 \cdot 5} \\ &= \sqrt[3]{8} \cdot \sqrt[3]{5} \\ &= 2\sqrt[3]{5}\end{aligned}$$

14.20 Simplify the expression: $\sqrt[3]{-16}\left(\sqrt[3]{4}\right)$.

Roots with even indices (including square roots) can't have a negative radicand—you won't see a negative sign under those radicals.

If a root with an odd index (cube roots, fifth roots, seventh roots, etc.) has a negative radicand, that negative sign may be moved outside of the radical. In this problem, $\sqrt[3]{-16} = -\sqrt[3]{16}$. Rewrite the expression, moving the negative sign out of the radicand.

$$\sqrt[3]{-16}\left(\sqrt[3]{4}\right) = -\sqrt[3]{16}\left(\sqrt[3]{4}\right)$$

Neither radicand in the product is a perfect cube, but 16 has perfect cube factor 8. Rewrite 16 as the product $8 \cdot 2$ and simplify.

$$\begin{aligned}&= -\sqrt[3]{8 \cdot 2}\left(\sqrt[3]{4}\right) \\ &= \left(-\sqrt[3]{8}\sqrt[3]{2}\right)\left(\sqrt[3]{4}\right) \\ &= -2\sqrt[3]{2}\left(\sqrt[3]{4}\right)\end{aligned}$$

The radical expressions share the same index (3), so the product of the roots is equal to the root of the product of the radicands.

$$=-2\sqrt[3]{2\cdot 4}$$
$$=-2\sqrt[3]{8}$$

The product of the radicands is a perfect cube; simplify the expression.

$$=-2(2)$$
$$=-4$$

You conclude that $\sqrt[3]{-16}\left(\sqrt[3]{4}\right)=-4$.

14.21 Simplify the expression: $\sqrt[3]{9}\left(\sqrt[3]{48}\right)$.

Simplify the radical expression $\sqrt[3]{48}$, noting that the radicand has perfect cube factor 8.

$$\begin{aligned}\sqrt[3]{9}\left(\sqrt[3]{48}\right)&=\sqrt[3]{9}\left(\sqrt[3]{8\cdot 6}\right)\\&=\sqrt[3]{9}\left(\sqrt[3]{8}\cdot\sqrt[3]{6}\right)\\&=\sqrt[3]{9}\cdot 2\cdot\sqrt[3]{6}\end{aligned}$$

It doesn't matter that 2 is stuck between the cube roots. As long as expressions are multiplied, you can move them around.

Calculate the product of the cube roots and simplify the expression.

$$=2\cdot\sqrt[3]{9}\cdot\sqrt[3]{6}$$
$$=2\sqrt[3]{9\cdot 6}$$
$$=2\sqrt[3]{54}$$

The radicand has perfect cube factor 27; simplify the radical expression.

$$=2\sqrt[3]{27\cdot 2}$$
$$=2\sqrt[3]{27}\cdot\sqrt[3]{2}$$
$$=2\cdot 3\cdot\sqrt[3]{2}$$
$$=6\sqrt[3]{2}$$

14.22 Simplify the expression: $\sqrt[3]{\dfrac{2{,}000}{216}}$.

The cube root of a quotient is equal to the quotient of the individual cube roots.

$$\sqrt[3]{\frac{2{,}000}{216}}=\frac{\sqrt[3]{2{,}000}}{\sqrt[3]{216}}$$

The radicand in the numerator has perfect cube factor 1,000; the radicand in the denominator is a perfect cube: $216=6^3$.

$$= \frac{\sqrt[3]{1,000 \cdot 2}}{\sqrt[3]{216}}$$
$$= \frac{\sqrt[3]{1,000} \cdot \sqrt[3]{2}}{\sqrt[3]{216}}$$
$$= \frac{10\sqrt[3]{2}}{6}$$

To reduce the fraction to lowest terms, divide the numerator value 10 and the denominator value 6 by their greatest common factor, 2.

$$= \frac{5\,\cancel{10}\sqrt[3]{2}}{3\,\cancel{6}}$$
$$= \frac{5\sqrt[3]{2}}{3}$$

Thus, $\sqrt[3]{\frac{2,000}{216}} = \frac{5\sqrt[3]{2}}{3}$. It is also correct to report the answer as $\frac{5}{3}\sqrt[3]{2}$.

Other Roots

Indices other than 2 and 3

14.23 Describe the method used to simplify roots other than square or cube roots by generalizing the techniques applied in Problems 14.1–14.22.

Consider the preceding problems in this chapter. Before you are asked to simplify square roots, Problem 14.2 directs you to construct a list of common perfect squares. This is a useful exercise, because it aids in the identification of perfect square factors. Similarly, in Problem 14.15, you list common perfect cubes before you are asked to simplify cube roots.

The same inverse relationship exists between all corresponding exponents and roots. For example, noting that $5^4 = 625$ (five to the fourth power equals 625) allows you to conclude that $\sqrt[4]{625} = 5$ (the fourth root of 625 equals 5).

14.24 Simplify the expression: $\sqrt[4]{81}$.

The expression $\sqrt[4]{81}$ asks "what number, if raised to the fourth power, is equal to 81?".

To simplify a fourth root, you need to identify factors of the radicand that are integers raised to the fourth power. Generate a short list of possible candidates by raising five consecutive positive integers to the fourth power, beginning with 2.

$$2^4 = 16$$
$$3^4 = 81$$
$$4^4 = 256$$
$$5^4 = 625$$
$$6^4 = 1,296$$

Because $3^4 = 81$, you conclude that $\sqrt[4]{81} = 3$.

14.25 Simplify the expression: $\sqrt[7]{2{,}187}$.

Construct a list of consecutive integers raised to the seventh power, beginning with 2.

There's no reason to start with 1, because 1 to any power equals 1. That doesn't really help you simplify radicals.

$$2^7 = 128$$
$$3^7 = 2{,}187$$

There's no need to generate a longer list, because raising additional integers to the seventh power quickly becomes unwieldy: $4^7 = 16{,}384$, $5^7 = 78{,}125$, and $6^7 = 279{,}936$. As the exponents grow larger, the number of candidate numbers you should identify will decrease. In this problem, because $3^7 = 2{,}187$, you conclude that $\sqrt[7]{2{,}187} = 3$.

14.26 Simplify the expression: $\sqrt[6]{4{,}096}$.

Raise consecutive integers to the sixth power, beginning with 2, until the list contains the radicand 4,096 or the list produces a number greater than the radicand.

If 4,096 is on the list, you're done. If you get a number larger than 4,096, then stop making the list and go back to see if it contains any factors of the radicand.

$$2^6 = 64$$
$$3^6 = 729$$
$$4^6 = 4{,}096$$

Because $4^6 = 4{,}096$, you conclude that $\sqrt[6]{4{,}096} = 4$.

14.27 Simplify the expression: $\sqrt[5]{1{,}215}$.

Determine whether the radicand 1,215 is equivalent to an integer raised to the fifth power by raising consecutive integers to the fifth power, beginning with 2.

$$2^5 = 32$$
$$3^5 = 243$$
$$4^5 = 1{,}024$$
$$5^5 = 3{,}125$$

Because $1{,}215 > 4^5$ but $1{,}215 < 5^5$, you can conclude that 1,215 is not equal to an integer raised to the fifth power; it is not worthwhile to calculate 6^5 or any higher base, because their values will be greater than 3,125. Therefore, your goal is to determine whether any of the fifth powers in the list you generated above are factors of 1,215.

Notice that $1{,}215 \div 243 = 5$. Rewrite the radicand as the product $5 \cdot 243 = 1{,}215$ and simplify.

$$\begin{aligned}\sqrt[5]{1{,}215} &= \sqrt[5]{243 \cdot 5} \\ &= \sqrt[5]{243} \cdot \sqrt[5]{5}\end{aligned}$$

Because $3^5 = 243$, it follows that $\sqrt[5]{243} = 3$. Substitute this value into the expression.

$$= 3 \cdot \sqrt[5]{5}$$
$$= 3\sqrt[5]{5}$$

You conclude that $\sqrt[5]{1,215} = 3\sqrt[5]{5}$.

14.28 Simplify the expression: $\sqrt[4]{7,500}$.

Problem 14.24 lists consecutive integers raised to the fourth power. Continue the list until it produces a value greater than or equal to the radicand 7,500.

$$\begin{array}{ll} 2^4 = 16 & 7^4 = 2,401 \\ 3^4 = 81 & 8^4 = 4,096 \\ 4^4 = 256 & 9^4 = 6,561 \\ 5^4 = 625 & 10^4 = 10,000 \\ 6^4 = 1,296 & \end{array}$$

Whereas 7,500 is not equal to an integer raised to the fourth power, it is evenly divisible by 5^4, as demonstrated below.

$$\begin{aligned} 7,500 \div 5^4 &= 7,500 \div 625 \\ &= 12 \end{aligned}$$

Rewrite the radicand as the product $625 \cdot 12$ and simplify.

$$\begin{aligned} \sqrt[4]{7,500} &= \sqrt[4]{625 \cdot 12} \\ &= \sqrt[4]{625} \cdot \sqrt[4]{12} \\ &= 5\sqrt[4]{12} \end{aligned}$$

14.29 Simplify the expression: $\sqrt[5]{192}\left(\sqrt[5]{3,125}\right)$.

Consult the list of integers raised to the fifth power presented in Problem 14.27. The radicand 3,125 appears on the list ($5^5 = 3,125$) and radicand 192 has a factor that appears on the list ($2^5 = 32$). Rewrite the expression, explicitly identifying factors that are integers raised to the fifth power, and simplify.

$$\begin{aligned} \sqrt[5]{192}\left(\sqrt[5]{3,125}\right) &= \sqrt[5]{32 \cdot 6}\left(\sqrt[5]{3,125}\right) \\ &= \sqrt[5]{32} \cdot \sqrt[5]{6}\left(\sqrt[5]{3,125}\right) \\ &= 2 \cdot \sqrt[5]{6} \cdot 5 \\ &= 2 \cdot 5 \cdot \sqrt[5]{6} \\ &= 10\sqrt[5]{6} \end{aligned}$$

If you're multiplying a string of numbers and radical expressions, multiply the numbers and radicals separately. In this problem, $2 \cdot 5 = 10$.

14.30 A student asked to simplify the expression $\sqrt[4]{81-16}$ responds with the incorrect answer below. Identify the error in the student's work.

$$\begin{aligned}\sqrt[4]{81-16} &= \sqrt[4]{81}-\sqrt[4]{16}\\ &= 3-2\\ &= 1\end{aligned}$$

The student has generalized the rule stating, "The root of a product is equal to the product of the roots" to subtraction, incorrectly concluding that the root of a difference is equal to the difference of the individual roots. Whereas you can split a product within a radical into two separate radicals $\left(\sqrt[4]{81\cdot 16}=\sqrt[4]{81}\cdot\sqrt[4]{16}\right)$ you cannot split a difference within a radical into two separate radicals $\left(\sqrt[4]{81-16}\neq\sqrt[4]{81}-\sqrt[4]{16}\right)$.

14.31 Simplify the expression: $\sqrt[4]{7{,}203}+\sqrt[4]{768}$.

Consult the list of consecutive integers raised to the fourth power presented in Problem 14.28. Note that the radicand 7,203 has factor $7^4 = 2{,}401$ that appears on the list; similarly, radicand 768 has factor $4^4 = 256$ that appears on the list. Rewrite the radicands as products that feature these factors.

$$\begin{aligned}\sqrt[4]{7{,}203}+\sqrt[4]{768} &= \sqrt[4]{2{,}401\cdot 3}+\sqrt[4]{256\cdot 3}\\ &= \left(\sqrt[4]{2{,}401}\cdot\sqrt[4]{3}\right)+\left(\sqrt[4]{256}\cdot\sqrt[4]{3}\right)\\ &= 7\sqrt[4]{3}+4\sqrt[4]{3}\end{aligned}$$

Problem 14.13 states that radical expressions cannot be added unless they have the same radicand. More specifically, both the radicands *and* the indices must match in order to calculate the sum or difference of radical expressions. In this problem, both roots have an index of 4 and both radicands are equal to 3, so the requirements are met and the sum may be calculated. Add coefficients 7 and 4 and multiply the sum by the shared radical expression.

$$\begin{aligned}&= (7+4)\sqrt[4]{3}\\ &= 11\sqrt[4]{3}\end{aligned}$$

Fractional Exponents

Changing fractional powers into roots

14.32 Rewrite the exponential expression as a radical expression: $7^{1/2}$.

A value raised to a fractional power may be rewritten as a radical expression. The numerator of the exponent becomes the power of the radicand, and the denominator of the exponential power becomes the index, as illustrated below.

$$x^{a/b}=\sqrt[b]{x^a}$$

Use this version when the radicand x is a large number and raising it to a power before taking the root would create an even bigger number.

In this problem, base 7 is raised to the $\frac{1}{2}$ power. The base becomes the radicand of the new expression, its exponent is the numerator of the fraction (1), and the index of the radical expression is the denominator of the fraction (2).

$$7^{1/2} = \sqrt[2]{7^1}$$

An index of two need not be explicitly written, as it indicates a square root. Similarly, an exponent of one may be omitted: $7^1 = 7$.

$$7^{1/2} = \sqrt{7}$$

Occasionally, an alternate form of the radical expression is preferable, in which the entire radical expression—not just the radicand—is raised to the a power: $x^{a/b} = \left(\sqrt[b]{x}\right)^a$. In this example, the alternative form produces the same result.

$$\begin{aligned} 7^{1/2} &= \left(\sqrt[2]{7}\right)^1 \\ &= \left(\sqrt{7}\right)^1 \\ &= \sqrt{7} \end{aligned}$$

14.33 Rewrite the exponential expression as a radical expression: $5^{2/3}$.

As Problem 14.32 explains, the base of the exponential expression (5) is the radicand of the new expression, the denominator of the exponent (3) is the index of the radical, and the numerator (2) is the power of the radicand.

$$\begin{aligned} 5^{2/3} &= \sqrt[3]{5^2} \\ &= \sqrt[3]{25} \end{aligned}$$

14.34 Rewrite the exponential expression as a radical expression: $10^{3/5}$.

The equivalent radical expression will have a radicand equal to the base (10), raised to a power equal to the numerator of the exponent (3), within a radical symbol with an index equal to the denominator of the exponent (5).

$$\begin{aligned} 10^{3/5} &= \sqrt[5]{10^3} \\ &= \sqrt[5]{1,000} \end{aligned}$$

Note: In Problems 14.35–14.36, you use two different formulas to convert the exponential expression $16^{5/4}$ into a radical expression.

14.35 Apply the formula $x^{a/b} = \sqrt[b]{x^a}$, as demonstrated in Problems 14.32–14.34, but do not simplify the expression. Explain why further simplification would be difficult, especially without the aid of a calculator.

The formula identified in this problem generates a fourth root with a radicand of 16 raised to the fifth power.

$$\begin{aligned} 16^{5/4} &= \sqrt[4]{16^5} \\ &= \sqrt[4]{1{,}048{,}576} \end{aligned}$$

The radicand $16^5 = 1{,}048{,}576$ is a rather large number, especially when simplifying the expression without the aid of a calculator. In fact, simply evaluating 16^5 is a time-consuming task. Given an number raised to a fractional exponent, the radicand of the equivalent exponential expression literally grows exponentially larger as the base and the numerator of the exponent increase.

Note: In Problems 14.35–14.36, you use two different formulas to convert the exponential expression $16^{5/4}$ into a radical expression.

14.36 Apply the formula $x^{a/b} = \left(\sqrt[b]{x}\right)^a$ and simplify the expression.

This formula calculates the fourth root of 16 *before* raising the radicand to the fifth power, which results in significantly smaller values.

$$16^{5/4} = \left(\sqrt[4]{16}\right)^5$$

Note that $2^4 = 16$, so it follows that $\sqrt[4]{16} = 2$. Replace the radical expression with its equivalent value in the equation above.

$$\begin{aligned} 16^{5/4} &= (2)^5 \\ &= 32 \end{aligned}$$

Therefore, $16^{5/4} = 32$.

Both of the exponential-to-radical expression formulas (from Problems 14.35 and 14.36) are useful in different circumstances. If you get stuck in a problem, restart and try the other formula.

14.37 Rewrite the exponential expression as a radical expression and simplify: $27^{4/3}$.

Apply the formula described in Problem 14.36, which evaluates a root before raising the radicand to a power.

$$27^{4/3} = \left(\sqrt[3]{27}\right)^4$$

Recall that $3^3 = 27$, so it follows that $\sqrt[3]{27} = 3$. Substitute the equivalent value into the equation above and simplify.

$$\begin{aligned} 27^{4/3} &= 3^4 \\ &= 81 \end{aligned}$$

Therefore, $27^{4/3} = 81$.

14.38 Rewrite as a sum of radical expressions and simplify: $4^{3/2}+5^{3/2}$.

You might be tempted to look for the "obvious" answer $9^{3/2}$, but that's wrong.

You may only add exponential expressions that have equal bases raised to equal powers. Therefore, you cannot calculate the sum of $4^{3/2}$ and $5^{3/2}$ immediately; they share the same exponent, but they contain different bases. Convert both into radical expressions and simplify.

$$\begin{aligned} 4^{3/2}+5^{3/2} &= \sqrt{4^3}+\sqrt{5^3} \\ &= \sqrt{64}+\sqrt{125} \end{aligned}$$

Both radicands have perfect square factors; identify them and simplify the expressions.

$$\begin{aligned} &= \sqrt{64}+\sqrt{25\cdot 5} \\ &= \sqrt{64}+\left(\sqrt{25}\cdot\sqrt{5}\right) \\ &= 8+5\sqrt{5} \end{aligned}$$

No further simplification is possible. However, there are other, equivalent solutions to this problem, based upon the method by which you convert the exponential expressions into radical expressions. The answer above assumes you apply the formula from Problem 14.35 to rewrite the exponential expressions; if you apply the formula from Problem 14.36, you get the solution below.

$$\begin{aligned} \left(\sqrt{4}\right)^3+\left(\sqrt{5}\right)^3 &= 2^3+\left(\sqrt{5}\right)^3 \\ &= 8+\left(\sqrt{5}\right)^3 \end{aligned}$$

While both $8+5\sqrt{5}$ and $8+\left(\sqrt{5}\right)^3$ are equally valid answers, the former is preferable; it contains no explicit exponents, so it arguably represents a better simplified solution.

14.39 Given the radical expression $\sqrt[6]{2^5}$, identify two different equivalent exponential expressions.

This problem directs you to reverse the procedure explored in Problems 14.32–14.38. Rather than beginning with an expression containing a fractional exponent, it begins with a radical expression and requires you to work backward. The same principles apply:

- The radicand is the base of the exponential expression.
- The exponent of the radicand is the numerator of the exponent.
- The index of the root is the denominator of the exponent.

In this problem, the radicand is 2, its exponent is 5, and the index of the root is 6, so $\sqrt[6]{2^5}=2^{5/6}$. To identify another exponential representation of the expression, evaluate 2^5 before undertaking the conversion.

$$\sqrt[6]{2^5}=\sqrt[6]{32}$$

The new radical expression will translate to an exponential expression with a base of 32, an exponential numerator of 1, and an exponential denominator of 6.

$$\sqrt[6]{32} = 32^{1/6}$$

14.40 Express the product as a simplified exponential expression: $\sqrt{5} \cdot \sqrt[3]{5}$.

While it is true that the product of two roots is equal to the root of the product of the radicands, that relationship only holds true if the indices of the roots are equal. In this problem, a square root is multiplied by a cube root, so you *cannot* conclude that $\sqrt{5} \cdot \sqrt[3]{5}$ is equal to either $\sqrt{5 \cdot 5}$ or $\sqrt[3]{5 \cdot 5}$.

In order to calculate the product, first rewrite the expression as a product of exponential expressions.

$$\begin{aligned} \sqrt{5} \cdot \sqrt[3]{5} &= \sqrt[2]{5^1} \cdot \sqrt[3]{5^1} \\ &= 5^{1/2} \cdot 5^{1/3} \end{aligned}$$

You don't have to write the 1 exponents or the 2 index, but it makes it easier to see where all of the numbers in the next line are coming from.

According to Problem 13.23, the product of two exponential expressions with the same base is equal to the shared base raised to the sum of the powers. In this problem, $5^{1/2}$ and $5^{1/3}$ both have a base of 5, so the product is 5 raised to the $\frac{1}{2} + \frac{1}{3}$ power.

$$= 5^{\frac{1}{2} + \frac{1}{3}}$$

Add the fractions by rewriting them in terms of the least common denominator 6.

$$\begin{aligned} &= 5^{\frac{3}{6} + \frac{2}{6}} \\ &= 5^{\frac{5}{6}} \end{aligned}$$

You conclude that $\sqrt{5} \cdot \sqrt[3]{5} = 5^{5/6}$.

14.41 Express the product as a simplified exponential expression: $\sqrt[3]{2} \cdot \sqrt[9]{4}$.

Like Problem 14.39, this expression represents the product of two roots with different indices. To multiply them, first rewrite the radical expressions as exponential expressions.

$$\begin{aligned} \sqrt[3]{2} \cdot \sqrt[9]{4} &= \sqrt[3]{2^1} \cdot \sqrt[9]{4^1} \\ &= 2^{1/3} \cdot 4^{1/9} \end{aligned}$$

Exponential expressions may be multiplied only when the bases are equal. The bases in this expression (2 and 4) are unequal, but because 4 is a power of 2, that complication is easily remedied. Note that $2^2 = 4$, so you can replace 4 with the exponential expression 2^2, which has the necessary base.

$$= 2^{1/3} \cdot \left(2^2\right)^{1/9}$$

According to Problem 13.26, if an exponential expression is, itself, raised to a power, you should raise the base of the expression to the product of the exponents. In this problem, 2 is raised to the power of 2, and that expression is raised to the $\frac{1}{9}$ power.

$$\begin{aligned} &= 2^{1/3} \cdot 2^{2\left(\frac{1}{9}\right)} \\ &= 2^{1/3} \cdot 2^{\frac{2}{1}\left(\frac{1}{9}\right)} \\ &= 2^{1/3} \cdot 2^{2/9} \end{aligned}$$

Now that the expressions share a common base, you can calculate the product by adding the exponents. To add the fractions, you must first rewrite them in terms of the least common denominator, 9.

$$\begin{aligned} &= 2^{\frac{1}{3}+\frac{2}{9}} \\ &= 2^{\frac{3}{9}+\frac{2}{9}} \\ &= 2^{5/9} \end{aligned}$$

You conclude that $\sqrt[3]{2} \cdot \sqrt[9]{4} = 2^{5/9}$.

14.42 Express the product as a simplified exponential expression: $\sqrt[4]{3}\left(\sqrt{\frac{1}{3}}\right)$.

Like Problems 14.40–14.41, this problem begins with an expression containing radicals with different indices. Rewrite the expression using exponential notation.

$$\sqrt[4]{3}\left(\sqrt{\frac{1}{3}}\right) = 3^{1/4}\left(\frac{1}{3}\right)^{1/2}$$

The bases of the exponential expressions must be equal before you can calculate the product. Recall that an exponent of –1 produces the reciprocal of a fraction. Specifically, Problem 13.32 states $3^{-1} = \frac{1}{3}$; the reciprocal of 3 is equal to $\frac{1}{3}$.

$$= 3^{1/4}\left(3^{-1}\right)^{1/2}$$

An exponential expression that is, itself, raised to a power is equal to the base raised to the product of the powers.

$$= 3^{1/4}\left(3^{-1[1/2]}\right)$$
$$= 3^{1/4}\left(3^{-1/2}\right)$$

The product of two exponential expressions with the same base is that base raised to the sum of the powers.

$$= 3^{\frac{1}{4}+\left(-\frac{1}{2}\right)}$$
$$= 3^{\frac{1}{4}-\frac{1}{2}}$$
$$= 3^{\frac{1}{4}-\frac{2}{4}}$$
$$= 3^{-\frac{1}{4}}$$

A simplified exponential expression should not contain a negative exponent.

$$= \left(\frac{1}{3}\right)^{1/4}$$

Chapter 15
VARIABLES

Letters that represent numbers

The principles explored throughout the preceding chapters are not solely applicable to real numbers. In fact, the true strength of mathematics lies in its abilities to identify unknown values and draw conclusions based on known data. In this chapter, you will explore the foundations of variables, begin to construct simple variable expressions, and calculate the values of those expressions when specific values are assigned to the variables within them.

The two most common directions for math problems are "simplify" (which you've done a lot of if you've worked through Chapters 1–14) and "solve for x." Before you can solve for x, however, you need to understand what role x, or any other variable, actually plays in an expression. If you're the philosophical type, you might wonder, "How can I represent that which is unknown by a concrete, known letter?" I wouldn't let it keep you up at night. For now, focus on learning the language of variables, because the familiarity you'll build in this chapter will be important in Chapters 16 and 17.

Types and Uses of Variables

Constants, unknowns, dependent, and independent

15.1 Identify the variables in the expression: $2x + 3y^2 - 1$.

If you divide the circumference of a circle by the diameter of the same circle, you always get the same decimal: $\pi = 3.141519....$ If you have no idea what I am talking about, the book gets to this in Chapter 19.

This expression is comprised of numbers, operators (that represent addition and subtraction), and variables. Variables are letters—in this case x and y—that represent numbers. For example, $2x$ is the product of the number 2 and some other real number. Rather than write "$2 \cdot$ unknown number," the variable x takes the place of the unknown value.

The variables most commonly used in algebra are x, y, z, w, a, b, c, d, and m; textbooks traditionally express variables italicized and in lowercase. Occasionally, Greek letters are used as variables. The most common Greek variable is π (spelled "pi" and pronounced the same as "pie") is a nonterminating, nonrepeating decimal that represents the quotient of the circumference of a circle and its diameter.

15.2 Describe the difference between variables and constants.

The word "variable" describes any letter that represents a numeric value. However, within an expression, some values vary less than others. A "constant" is a variable whose value is either permanently or temporarily fixed, even if the value is only fixed for the duration of a single problem. This is a very subtle distinction, and many textbooks do not differentiate between them; further, not all texts agree on the distinction. However, this is a suitable working definition, especially in the context of basic algebra.

Consider the following problem as an example: "Given the expression $4ax$, assume that $a = 5$ and evaluate the expression for $x = 1$, $x = 2$, and $x = 3$." The expression $4ax$ contains two variables (a and x), but one of those variables (a) is assigned the same value for the entire problem, whereas the other variable (x) truly varies—it is assigned three different values. In this context, though a and x are considered variables, a is also considered a constant, because it might as well be the fixed, constant number 5, at least in this problem. To help distinguish constants, they are often assigned to variables such as a, b, c, and d, whereas more generic variables are often identified as x, y, or z.

To make things even less clear, plain old numbers are also called constants. For example, in the expression "$xy + 2$," the number 2 is called a constant because its value doesn't change.

15.3 The statement below describes the slope-intercept formula, which is used to construct equations whose graphs are straight lines. The formula includes four variables. Based on the statement, determine which two variables are constants and explain your answer.

Lines can be expressed as equations in slope-intercept form, the equation $y = mx + b$. Every line has a specific, fixed slope m and y-intercept b. Given those values, for any real number x, you can calculate a corresponding y-value.

Even if you have no knowledge of lines and the formulas used to construct their equations, this statement contains all the information you need; the key word is "fixed." Every line has a specific slope (which describes the steepness of the line) and a specific y-intercept (which describes an important intersection point on the graph).

Different lines have different slopes and y-intercepts, but all lines have one fixed slope and one fixed y-intercept. Because the variables m and b, which represent those fixed values, do not change if you are discussing a single line, they are constants, whereas x and y fall under the less specific category of "variables."

15.4 Translate the expression $x + 9$ into a phrase that describes its value.

The expression $x + 9$ represents the sum of an unknown value x and the whole number 9. If you add 9 to an unknown value, the result is "nine more than some number x," which is usually written "nine more than a number."

15.5 Translate the expression $7y$ into a phrase that describes its value.

The expression $7y$ is read "the product of seven and a number." It is also correct to translate the expression as "seven times a number" or "a number times seven."

15.6 Translate the expression $2w - 5$ into a phrase that describes its value.

This expression multiplies the variable w by two and then subtracts five from that product. The order of operations applies to expressions containing variables, which is why multiplication must be completed before subtraction. There are many correct answers to this problem, but a generally preferred response is "five less than twice a number." Other examples of correct responses are listed below:

- Twice a number less five
- Five less than a number doubled
- The sum of twice a number and negative five
- Two times a number minus five

The "less" and "less than" phrases are tricky. "Two less THAN a number" means $x - 2$, but "two less a number" is $2 - x$. The order in which you subtract matters: $x - 2$ and $2 - x$ usually aren't equal.

Translating between English phrases and mathematical expressions is explored further in Problems 15.9–15.20.

If you assign a value (usually by substituting something in), it's an independent variable. If the value is the result of simplifying after substitution, it's a dependent variable.

15.7 Describe the relationship between dependent and independent variables.

Dependent and independent variables are typically paired with one another, appearing in the same equation or formula. The value of a dependent variable is dictated by the value of an independent variable. You typically "substitute" a real number value for one variable in an equation; that value replaces the independent variable. Once the formula or equation is simplified, the net result is a value for the remaining, dependent variable.

15.8 The formula for the area of a circle is $A = \pi r^2$. For any radius value r you choose, the product of r^2 and the fixed value $\pi = 3.141519...$ is equal to the area A of the circle. This formula contains three variables: one constant, one dependent, and one independent. Identify each and explain your answer.

The word "fixed" identifies the constant in the formula: π, which is a fixed value not only for this problem but for mathematics in general. The independent variable is the variable to which you assign a value, the variable you "substitute into," replacing it with a real number. The problem states, "For any radius value r you choose," which indicates that you are replacing r with a value of your choice. Thus, r is the independent variable.

Once a value is assigned to r, you can square it and multiply by π; that product is equal to A. The value of A depends entirely upon the value you choose for r, so A is the dependent variable.

Constructing Variable Expressions

Translate English to math

15.9 Translate the phrase below into a mathematical expression written in terms of x.

Six more than a number

You are directed to write an expression in terms of x, which means the unknown number referenced in the expression should be represented by x. Six more than some number x is equivalent to the mathematical expression $x + 6$.

To check your answer, assign a real number value to x. For example, assume the "number" in the expression is one. Six more than one is seven. When you replace x with the value you assigned (1), the result is 7, as demonstrated below.

$$\begin{aligned} x+6 &= 1+6 \\ &= 7 \end{aligned}$$

15.10 Translate the phrase below into a mathematical expression written in terms of x.

Eleven less than a number

"Eleven less than a number" is best represented by the expression $x - 11$. Like Problem 15.9, this concept is more easily understood in the context of a real number example. Choose a number and substitute it into the phrase; for instance, "Eleven less than fifteen." The result is four: $15 - 11 = 4$.

If the original phrase, itself, did not cue you that you needed to perform subtraction, the real number example should. To calculate 11 less than a number x, you should subtract 11 from x.

15.11 Translate the phrase below into a mathematical expression written in terms of x.

The difference of a number and eight

The word "difference" indicates subtraction should be performed. Unlike addition and multiplication, the order in which you subtract is important, and it is dictated by the order in which the quantities appear in the expression. The phrase, "The difference of A and B," should be translated $A - B$; the first quantity named in the phrase (A) is the first quantity written in the subtraction problem.

Phrases that indicate subtraction include difference, less, less than, fewer than, reduced by, and (the obvious one) minus.

In this problem, "a number" appears before "eight," so the phrase translates mathematically as $x - 8$.

15.12 Translate the phrase below into a mathematical expression written in terms of x.

Nineteen less a number

In Problem 15.10, the phrase, "Eleven less than a number" translates into $x - 11$. In order to calculate eleven less than some value, you must subtract eleven from that value. This problem contains a phrase that is slightly different; it omits the word "than." If the problem had read, "Nineteen less *than* a number," the correct answer would have been $x - 19$. However, the omission of the word "than" reverses the order in which you subtract.

The phrase "nineteen less a number" suggests that you begin with an initial value of nineteen and remove some value from it. The result is "less" because you subtracted "a number." Hence, it is represented mathematically as $19 - x$.

15.13 Translate the phrase below into a mathematical expression written in terms of x.

One-fifth of a number

While it is not an absolute rule without exception, the word "of" generally represents multiplication, especially in the context of fractions. In this problem, "one-fifth of a number" is equal to the fraction one-fifth multiplied by the number: $\frac{1}{5}x$. If you wish to express the quantity as a single fraction, divide x by 1 to express it as a fraction and multiply, as demonstrated below.

$$\begin{aligned}\frac{1}{5}x &= \frac{1}{5}\cdot\frac{x}{1}\\ &= \frac{1\cdot x}{5\cdot 1}\\ &= \frac{x}{5}\end{aligned}$$

Both $\frac{1}{5}x$ and $\frac{x}{5}$ are equally valid answers.

15.14 Translate the phrase below into a mathematical expression written in terms of x.

The product of six and one less than a number

The word "product" in the phrase indicates multiplication. However, "less than" also indicates the presence of subtraction. Read the phrase carefully to determine which operations are applied to which quantities. The phrase begins with, "The product of six and…," so you deduce that 6 is multiplied by the value following the word "and"; that phrase is "one less than a number."

$$6 \cdot \text{(one less than a number)}$$

According to Problem 15.10, "one less than a number" should be translated $x-1$. Substitute this value for the parenthetical phrase above.

$$6(x-1)$$

Check your answer by translating it back into an English phrase. The expression $6(x-1)$ is read, "Six times one less than some number," which is equivalent to the original phrase given in the problem.

15.15 Translate the phrase below into a mathematical expression written in terms of x.

Four more than twice a number

Here are some multiplication trigger words: times, product, of (see Problem 15.13), twice, double, triple, quadruple, etc.

"More than" indicates that the answer will contain addition. "Twice" is a specific indicator of multiplication—twice a number is equal to that number multiplied by two. There are two operations present in this phrase, so both must appear in the mathematical expression. Reread the phrase carefully: "Four more than" indicates you should add four to the phrase that follows, "Twice a number."

$$(\text{Twice a number}) + 4$$

Twice some number x is written $2 \cdot x$ or $2x$. Substitute this into the parenthetical phrase above.

$$2x + 4$$

Therefore, four more than twice a number is translated mathematically as $2x + 4$.

15.16 Translate the phrase below into a fraction written in terms of x.

Given a real number, the quotient of one more than that number and one less than the number

The word "quotient" indicates division. In this problem, you are dividing two quantities: one more than some number and one less than that original number. One more than a number is written $x + 1$; one less than a number is written $x - 1$.

Like subtraction, the order in which a division problem states its quantities is important. The quotient of A and B is equal to $A \div B$; the first value in the phrase is the dividend and the second value is the divisor. This problem states that the sum is divided by the difference.

$$\frac{x+1}{x-1}$$

Another correct answer is $\frac{1+x}{x-1}$. You can reverse the numbers in the numerator, but not in the denominator.

If the problem had not specifically instructed you to express the quotient as a fraction, the answer $(x + 1) \div (x - 1)$ would also have been acceptable—as long as both sets of parentheses are included.

15.17 Translate the phrase below into a mathematical expression written in terms of x.

The square of the sum of a number and three

Key words that indicate addition: sum, more, more than, in addition to, increased by, and (of course) plus.

The phrase, "The square of" indicates that the phrase that immediately follows should be raised to the second power; thus, you should square "the sum of a number and three."

$$(\text{the sum of a number and three})^2$$

The word "sum" cues you to add "a number" and "three." Replace the phrase above with its mathematical equivalent.

$$(x+3)^2$$

15.18 Translate each of the phrases below into a mathematical expression written in terms of x:

(a) The opposite of the square of a number
(b) The square of the opposite of a number

(a) You are asked to calculate the opposite of something; specifically, the problem asks you to calculate the opposite of the square of a number: x^2. (Recall that squaring means raising something to the second power.) To calculate the opposite of a number, multiply by -1. Thus, the opposite of x^2 is $-1x^2$ or, more simply, $-x^2$.

(b) This phrase begins with "the square of," so the entire quantity that is described following those words should be enclosed in parentheses and squared.

$$(\text{the opposite of a number})^2$$

The opposite of some number x is the product of that number and -1; in other words, the opposite of x is $(-1) \cdot x = -x$.

$$(-x)^2$$

Notice that the square of a number, x^2, and the square of its opposite, $(-x)^2$, are both equal to x^2. A negative number times itself is a positive number.

$$(-x)^2 = (-x)(-x) = x^2$$

Though you can simplify the expression $(-x)^2$ to get x^2, the problem specifically asks you to express "The square of the opposite of a number" mathematically, so the correct answer is $(-x)^2$.

15.19 Translate the phrase below into a mathematical expression written in terms of x.

Nine less than the square of five more than a number

Analyze the expression, identify all three operations that are present, and determe how each affects the mathematical expression. The first phrase, "nine less than," tells you to subtract nine from the entire phrase that follows.

(the square of five more than a number) – 9

The phrase "the square of" directs you to raise the proceeding quantity (five more than a number) to the second power.

(five more than a number)^2 – 9

Finally, the remaining parenthetical phrase translates mathematically as $x + 5$. Substitute it into the expression above.

$$(x+5)^2 - 9$$

15.20 Translate the phrase below into a mathematical expression.

The quotient of triple a number and the cube of another number

A quotient is a division problem. The dividend consists of the words between "quotient of" and the word "and," which marks the beginning of the divisor. In this problem, the dividend is "triple a number" and the divisor is "the cube of another number."

There could be multiple ANDs in a problem, but it shouldn't be hard to tell which separates what you're dividing INTO and what you're dividing BY.

(triple a number) ÷ (the cube of another number)

Note that "another number" is used to describe the divisor, so you should use different variables for each expression. To triple a number, multiply it by three. To cube a number, raise it to the third power.

$$(3x) \div y^3$$

You do not have to use x and y in your answer—just ensure that the variables are different. It is also important that $3x$ be enclosed in parentheses; the answer $3x \div y^3$ is technically correct here, but parentheses are recommended when quantities are multiplied and divided. To avoid using parentheses altogether, you may choose to express your answer as a fraction.

$$\frac{3x}{y^3}$$

Evaluating Variable Expressions

Substitute values into variables

15.21 Identify the algebraic property that allows you to evaluate expressions given real number values for variables. Explain your answer using the expression $x+1$ as an example; evaluate the expression given $x=2$.

The substitution property of equality allows you to replace one value with an equivalent value. In terms of variable expressions, you will normally replace one or more variables with real number values. For instance, you may be asked to evaluate the expression $x+1$ if $x=2$. According to the substitution property, you can replace x in the expression $x+1$ with its equivalent value, 2.

$$\begin{aligned} x+1 &= 2+1 \\ &= 3 \end{aligned}$$

You conclude that $x+1$ is equal to 3 if you assume that $x=2$.

15.22 Evaluate $4x+9$ given $x=5$.

According to the substitution property of equality, you can replace x in the expression $4x+9$ with the value 5, because $x=5$.

If you need to review the order of operations, flip back to Chapter 4.

$$4x+9=4(5)+9$$

Simplify the expression. According to the order of operations, you must multiply before you add.

$$\begin{aligned} 4(5)+9 &= 20+9 \\ &= 29 \end{aligned}$$

Therefore, $4x+9=29$ if $x=5$.

15.23 Evaluate $6x^2y$ given $x=-1$ and $y=2$.

Replace x with -1 and y with 2 in the expression. Note that you are squaring x, so -1 should be placed in parentheses. It is usually beneficial to surround numbers with parentheses when they are substituted into an expression, in this case resulting in $6(-1)^2(2)$ rather than $6-1^22$. The parentheses do not add complexity, but they do produce a more meaningful and understandable expression.

The expression contains x^2, so whatever x is, that number needs to be multiplied by itself: $(-1)(-1)=+1$

$$6x^2y=6(-1)^2(2)$$

Exponents must be evaluated before you multiply.

$$=6(1)(2)$$

Multiply from left to right to complete the problem.

$$\begin{aligned} &= 6(2) \\ &= 12 \end{aligned}$$

You conclude that $6x^2y=12$ if $x=-1$ and $y=2$.

Note: Problems 15.24–15.25 refer to the expression $x - xy - 4z$.

15.24 Evaluate the expression given $x = 4$, $y = 3$, and $z = 9$.

Substitute x, y, and z into the expression.

$$x - xy - 4z = (4) - (4)(3) - 4(9)$$

The second and third terms, as you count from left to right, contain products. Multiplication precedes subtraction in the order of operations.

$$\begin{aligned} &= 4 - (12) - (36) \\ &= 4 - 12 - 36 \\ &= -8 - 36 \\ &= -44 \end{aligned}$$

Thus, $x - xy - 4z = -44$, if $x = 4$, $y = 3$, and $z = 9$.

Note: Problems 15.24–15.25 refer to the expression $x - xy - 4z$.

15.25 Evaluate the expression given $x = -6$, $y = 0$, and $z = -5$.

Substitute the variables into the expression and simplify. Note that $y = 0$, so the middle term $(-xy)$ will be equal to zero and is effectively eliminated from the expression.

$$\begin{aligned} x - xy - 4z &= (-6) - (-6)(0) - 4(-5) \\ &= -6 - 0 + 20 \\ &= -6 + 20 \\ &= 14 \end{aligned}$$

$-4(-5) = +20$, because the product of two negatives is a positive.

15.26 Evaluate the expression $\frac{2x - y}{y^2 - x}$ given $x = 7$ and $y = 5$.

Substitute the values of x and y into the expression, then simplify the numerator and denominator independently.

$$\begin{aligned} \frac{2x - y}{y^2 - x} &= \frac{2(7) - (5)}{(5)^2 - (7)} \\ &= \frac{14 - 5}{25 - 7} \\ &= \frac{9}{18} \end{aligned}$$

Divide the numerator and denominator by their greatest common factor (9) to reduce the fraction to lowest terms.

$$\begin{aligned} &= \frac{9 \div 9}{18 \div 9} \\ &= \frac{1}{2} \end{aligned}$$

Note: Problems 15.27–15.28 refer to the expression $5|xy^3|$.

15.27 Evaluate the expression given $x = 8$ and $y = -\frac{1}{2}$.

Substitute the values of x and y into the expression. Grouping symbols—in the form of absolute value bars—are present, so the expression within must be simplified first.

$$\begin{aligned}5|xy^3| &= 5\left|(8)\left(-\frac{1}{2}\right)^3\right| \\ &= 5\left|(8)\left(-\frac{1}{8}\right)\right| \\ &= 5\left|\left(\frac{8}{1}\right)\left(-\frac{1}{8}\right)\right| \\ &= 5\left|-\frac{8}{8}\right| \\ &= 5|-1|\end{aligned}$$

The absolute value of a negative number is a positive number: $|-1| = 1$.

$$\begin{aligned}&= 5(1) \\ &= 5\end{aligned}$$

Note: Problems 15.27–15.28 refer to the expression $5|xy^3|$.

15.28 Evaluate the expression given $x = -2$ and $y = -3$.

Replace x with -2 in the expression, replace y with -3, and simplify.

$$\begin{aligned}5|xy^3| &= 5|(-2)(-3)^3| \\ &= 5|(-2)(-27)| \\ &= 5|54|\end{aligned}$$

The absolute value of any positive number is that positive number.

$$\begin{aligned}&= 5(54) \\ &= 270\end{aligned}$$

15.29 Evaluate the expression $\sqrt{y^2 - x^2}$ given $x = -4$ and $y = 5$.

The squares of both positive and negative numbers are positive numbers.

$$\begin{aligned}\sqrt{y^2 - x^2} &= \sqrt{(5)^2 - (-4)^2}\\ &= \sqrt{25 - (16)}\\ &= \sqrt{25 - 16}\\ &= \sqrt{9}\\ &= 3\end{aligned}$$

Exponents come before multiplication: $-(-4)^2 = -(+16)$. That gives you opposite adjacent signs $-+$, which turn into a negative sign: $-(+16) = -16$.

15.30 Evaluate the expression $\sqrt[3]{(1-w)(3w+1)}$ given $w = 5$.

Substitute $w = 5$ into the expression. There are three groups present, two defined by parentheses and a larger group that is the radicand—everything beneath the radical symbol. Begin by simplifying the parenthetical groups independently.

$$\begin{aligned}\sqrt[3]{(1-w)(3w+1)} &= \sqrt[3]{(1-(5))(3(5)+1)}\\ &= \sqrt[3]{(1-5)(15+1)}\\ &= \sqrt[3]{(-4)(16)}\\ &= \sqrt[3]{-64}\end{aligned}$$

Radical expressions with even indices cannot have negative radicands. However, this expression has an odd index (3), so the negative sign may be moved outside of the radical.

$$= -\sqrt[3]{64}$$

Because $4^3 = 64$, you conclude that $\sqrt[3]{64} = 4$.

$$\begin{aligned}&= -(4)\\ &= -4\end{aligned}$$

Therefore, $\sqrt[3]{(1-w)(3w+1)} = -4$ if $w = 5$.

15.31 Evaluate the expression $-x^2 + 1$ given $x = 4$.

Substitute $x = 4$ into the expression.

$$-x^2 + 1 = -(4)^2 + 1$$

Notice that only 4 is squared in the expression, not -4.

$$\begin{aligned}&= -(16) + 1\\ &= -16 + 1\\ &= -15\end{aligned}$$

15.32 Evaluate the expression $(5x)^{1/2} - y^{1/3}$ given $x = 20$ and $y = 16$.

Substitute the values of x and y into the expression.

$$\begin{aligned}(5x)^{1/2} - y^{1/3} &= (5\cdot 20)^{1/2} - (16)^{1/3}\\ &= 100^{1/2} - 16^{1/3}\end{aligned}$$

Apply the formula presented in Problem 14.32 to convert the exponential expressions into radical expressions.

$$= \sqrt{100} - \sqrt[3]{16}$$

The radicand of the square root is a perfect square; the radicand of the cube root has perfect cube factor 8.

$$\begin{aligned}&= \sqrt{100} - \sqrt[3]{8\cdot 2}\\ &= 10 - \sqrt[3]{8}\cdot\sqrt[3]{2}\\ &= 10 - 2\sqrt[3]{2}\end{aligned}$$

The answer $10 - 2^{4/3}$ is also correct, because:

$$\begin{aligned}2\sqrt[3]{2} &= 2^1 \cdot 2^{1/3}\\ &= 2^{1+(1/3)}\\ &= 2^{4/3}\end{aligned}$$

15.33 Evaluate the expression $2y^{3/4}$ given $y = 81$.

Substitute the value of y into the expression, rewrite the exponential expression as a radical expression, and simplify.

$$\begin{aligned}2y^{3/4} &= 2(81)^{3/4}\\ &= 2\left(\sqrt[4]{81}\right)^3\\ &= 2(3)^3\end{aligned}$$

Exponents must be evaluated before multiplication.

$$\begin{aligned}&= 2(27)\\ &= 54\end{aligned}$$

Anatomy of an Algebraic Expression

Coefficients, like, and unlike terms

15.34 Identify the terms in the expression: $3x^2 + 11x - 5$.

Terms are little clumps of numbers, variables, or numbers multiplied by variables. When you are adding two things together or subtracting one thing from another, those things are called terms.

The terms in an expression are separated by the operations of addition and subtraction. They typically consist of a number or variable, may be a product of a number *and* a variable, and may include exponents. This expression consists of three terms: $3x^2$, $11x$, and -5.

15.35 Identify the terms in the expression: $19x^5 + 6x^4 \cdot y^3 - 6xy$.

As Problem 15.34 explains, the terms in an expression are separated by addition or subtraction symbols. Thus, $6x^4$ and y^3 are *not* separate terms because they are multiplied; the term containing that product is best written $6x^4y^3$. There are a total of three terms in this expression: $19x^5$, $6x^4y^3$, and $-6xy$.

15.36 Construct two expressions containing x, y, and z, one in which the variables are terms and one in which the variables are factors.

According to Problems 15.34–15.35, terms of an expression are separated by addition or subtraction. The following expressions are possible solutions to the first part of this problem—they both contain terms x, y, and z.

$$x + y + z \qquad y + x + z + 11$$

Notice that the right expression above contains an additional term, the constant 11. This does not violate the requirements of the problem, which only require that the expressions contain the terms x, y, and z.

Whereas terms are numbers or variables separated by addition or subtraction, factors are numbers or variables that are separated by multiplication. Hence, an expression that has factors x, y, and z must contain the product of the variables. The following expressions are possible solutions to the second part of the problem—they both contain factors x, y, and z.

$$xyz \qquad -3xyz$$

Note: Problems 15.37–15.38 refer to the expression $7w^6 + 2w^4 - w + 9$.

15.37 Identify the coefficient(s) in the expression.

Coefficients are the real numbers in a term that are multiplied by variables. This expression contains four terms, three of which contain variables: $7w^6$, $2w^4$, and $-w$. The numbers preceding w^6, w^4, and w, respectively, are the coefficients of those terms: 7, 2, and –1.

When there's no number in front of a variable, its coefficient is 1. When there's only a negative sign in front of a variable (like –w in this problem), the coefficient is –1.

Note: Problems 15.37–15.38 refer to the expression $7w^6 + 2w^4 - w + 9$.

15.38 Identify the constant(s) in the expression.

As Problem 15.2 explains, the word "constant" may either refer to a variable with a fixed value or a real number that appears in an expression. In this problem, no variables are assigned values, so the word "constant" refers to the real number 9, which (unlike the other terms in the expression) is unaffected by the unknown value w and thus remains constant for all values of w.

15.39 Identify the like terms in the expression: $14x^2 + 4x - 6x^2 + 2$.

Like terms contain matching variables raised to identical powers. For example, $14x^2$ and $4x$ are *not* like terms. Although they both contain the same variable (x), that variable is raised to the second power in one term and the first power in the other.

Notice that terms $14x^2$ and $-6x^2$ contain the same variable (x) raised to the same power (2). Therefore, $14x^2$ and $-6x^2$ are like terms.

15.40 Identify the like terms in the expression: $-3x^2 - 10x^6y + yx^6 + 5$.

As Problem 15.39 explains, like terms contain the exact same variable (or variables) raised to identical powers. Notice that terms $-10x^6y$ and yx^6 both contain the same variables (x^6 and y) and each variable is raised to the same corresponding power (both x's are raised to the sixth power and both y's are raised to the first power). Although the variables are written in a different order in each term, they are still like terms, according to the commutative property of multiplication.

Which says that you can multiply in any order you want and you get the same thing:
$2 \cdot 6 \cdot 9 = 9 \cdot 2 \cdot 6$

Chapter 16

SIMPLIFYING VARIABLE EXPRESSIONS

Add, subtract, and multiply things that have variables

Basic math and pre-algebra topics often follow a predictable pattern, because the sequence of topics is a reliable scaffold for the construction of algebraic understanding. As the preceding chapter introduced variables and variable expressions, important new concepts, this chapter will serve to contextualize those concepts via the same medium used for whole numbers, fractions, decimals, percentages, exponential expressions, and radical expressions: the application of binary operations.

That very wordy paragraph is basically an apology. In far more words than necessary it says, "As soon as you learn something new (in this case, Chapter 15 talked about expressions that have x's and y's in them), the first thing you always have to do is add, subtract, multiply, and divide them. It isn't exciting, so sorry."

One piece of good news: You can forget about division, because it's a little too tricky for now. Focus on the other three operations.

Addition and Subtraction

Only combine like terms

16.1 Simplify the expression: $2x^3 + 5x^3$.

This expression contains two terms, $2x^3$ and $5x^3$. As first explained in Problem 15.34, two terms are deemed "like terms" if they contain the same variables raised to the same powers. In this example, both terms contain the same variable (x) raised to the same exponent (3), so $2x^3$ and $5x^3$ are like terms.

Distinguishing between like and unlike terms is important when simplifying expressions, for one specific reason: You can only add or subtract like terms. Unlike terms cannot be combined via addition and subtraction.

This is not true for multiplication (or division). You can multiply unlike terms together just as easily as like terms—there's no restriction on products (or quotients).

To add the like terms, add their coefficients (2 and 5, respectively) and multiply by the shared variable.

$$2x^3 + 5x^3 = 7x^3$$

This may feel familiar, as it is very similar to the process used to add and subtract radical expressions (outlined in Problem 14.13). To calculate a sum or difference of radical expressions, the radicands and indices of each term had to be equal.

16.2 Simplify the expression: $8x^2 + 4x$.

Because $8x^2$ and $4x$ are unlike terms (they share the same variable, x, but it is not raised to the same power in each term), they cannot be added together. Hence, you cannot simplify the expression any further; $8x^2 + 4x$ is the simplest form of the expression.

Trick question! You can't simplify it!

16.3 Simplify the expression: $(4x^2 + 10x) + (11x^2 + 2x)$.

This expression is split into two groups, each in a separate set of parentheses. However, the terms and the groups are added together, so the terms can be regrouped and reordered without changing the value of the expression.

It is helpful—though not required—to rewrite the expression, grouping pairs of like terms separately.

$$(4x^2 + 10x) + (11x^2 + 2x) = (4x^2 + 11x^2) + (10x + 2x)$$

Now the parentheses define two groups of like terms, one pair that contains x^2 and one pair that contains x. Add the pairs of like terms independently, by calculating the sum of the coefficients of the x^2-terms ($4 + 11 = 15$) and the sum of the coefficients of the x-terms ($10 + 2 = 12$).

$$= 15x^2 + 12x$$

16.4 Simplify the expression: $(15x + 2y) + (7y + 8x) + (3x + 9 + 4y)$.

There are three categories of like terms in this expression: terms containing x, terms containing y, and constants. Regroup all seven terms into groups of like terms and calculate the sum of each group. Note that the expression contains only one constant, so it appears in a group of its own.

Constants are terms that don't contain any variables at all, like 9.

$$\begin{aligned}(15x+2y)+(7y+8x)+(3x+9+4y) &= (15x+8x+3x)+(2y+7y+4y)+(9)\\ &= (23x+3x)+(9y+4y)+(9)\\ &= 26x+13y+9\end{aligned}$$

16.5 Simplify the expression: $17x^4 - 9x^4 - 3x^2 + 4x^2$.

"Subtracting terms" and "adding negative terms" refer to the same technique, which is very similar to the technique used to add terms (described in Problems 16.1–16.4). You can subtract terms only when they share the same variables raised to the same exponents—when they are like terms. Furthermore, you once again combine the coefficients of the like terms, as dictated by the signs.

Rewrite the expression, grouping the x^4-terms and x^2-terms separately.

$$17x^4 - 9x^4 - 3x^2 + 4x^2 = (17x^4 - 9x^4) + (-3x^2 + 4x^2)$$

Notice that an addition symbol is placed between the groups of like terms, despite the fact that $3x^2$ is a negative term. Combine the coefficients of the pairs of like terms: $17 - 9 = 8$ and $-3 + 4 = 1$.

$$= 8x^4 + 1x^2$$

It is not necessary to write coefficients of 1. However, it is important to remember that terms without an explicit coefficient are understood to have a coefficient of 1.

$$= 8x^4 + x^2$$

16.6 Simplify the expression: $(9x^2 + 5y^2) + (x^2 - y^2)$.

Rewrite the expression, grouping like terms.

$(9x^2 + 5y^2) + (x^2 - y^2) = (9x^2 + x^2) + (5y^2 - y^2)$

Although the x^2 and $-y^2$ terms have no explicit coefficients in the expression, Problem 16.5 reminds you that such terms have implied coefficients of 1. Combine the coefficients of the like terms: $9 + 1 = 10$ and $5 - 1 = 4$.

Actually, x^2 has a coefficient of 1, but $-y^2$ is negative, so it has a coefficient of -1.

$$= 10x^2 + 4y^2$$

16.7 Simplify the expression: $(5x - 9xy - 4y) + (2xy - 8y)$.

Like terms contain the exact same variable or variables, raised to the same corresponding powers. In this expression, $-4y$ and $-8y$ are like terms because they both contain y^1. Notice that $-9xy$ and $2xy$ are also like terms, because they contain the same variables (x and y) raised to the same powers (1 and 1, respectively). Rewrite the expression, grouping like terms.

$$(5x - 9xy - 4y) + (2xy - 8y) = (5x) + (-9xy + 2xy) + (-4y - 8y)$$

Combine the coefficients of the like terms: $-9 + 2 = -7$ and $-4 - 8 = -12$.

$$= 5x - 7xy - 12y$$

16.8 Simplify the expression: $(5x)^2 - 8x^2$.

Although $5x$ and $-8x^2$ are not like terms—because the shared variable x is raised to different powers in each term—the order of operations prohibits you from combining the terms until you apply the exponent 2 to $5x$.

$$\begin{aligned}(5x)^2 - 8x^2 &= 5^2 \cdot x^2 - 8x^2 \\ &= 25x^2 - 8x^2\end{aligned}$$

Now that the first term has been squared, the resulting expression contains two like terms, which can be combined.

$$= 17x^2$$

16.9 Simplify the expression: $4x(xy) - 2y(yx) + 6y(x^2)$.

This problem requires you to simplify the individual terms before you can simplify the expression as a whole. Recall that the product of two exponential expressions is equal to the shared base raised to the sum of the powers.

For example, to calculate the product $4x(xy)$, start by writing in the implied powers: $4x^1(x^1y^1)$. Then, add the powers when the bases match. In this case, $4x^1$ and x^1y^1 both contain an x, so the product contains $x^{1+1} = x^2$.

$$\begin{aligned}4x(xy) - 2y(yx) + 6y(x^2) &= 4 \cdot x \cdot x \cdot y - 2 \cdot y \cdot y \cdot x + 6 \cdot y \cdot x^2 \\ &= 4x^{1+1}y - 2y^{1+1}x + 6yx^2 \\ &= 4x^2y - 2y^2x + 6yx^2\end{aligned}$$

Convention dictates that multiple variables within a term should be written alphabetically. Thus, $-2y^2x = -2xy^2$ and $6yx^2 = 6x^2y$.

$$= 4x^2y - 2xy^2 + 6x^2y$$

Two of the three terms in the expression are like terms: $4x^2y$ and $6x^2y$ both contain x raised to the second power and y raised to the first power. Regroup like terms and combine their coefficients.

$$\begin{aligned}&= (4x^2y + 6x^2y) - 2xy^2 \\ &= 10x^2y - 2xy^2\end{aligned}$$

16.10 Simplify the expression: $xy(-10y) - 3xy(-4x^2) + 7xy^2 + 12x^3y$.

The first term multiplies y and y to produce y^2; the second term multiplies x and x^2 to produce x^3.

$$xy(-10y) - 3xy(-4x^2) + 7xy^2 + 12x^3y = -10xy^2 + 12x^3y + 7xy^2 + 12x^3y$$

Simplify the expression by combining the two pairs of like terms.

$$\begin{aligned} &= \left(-10xy^2 + 7xy^2\right) + \left(12x^3y + 12x^3y\right) \\ &= -3xy^2 + 24x^3y \end{aligned}$$

The Distributive Property Revisited

One thing times a bunch of things

16.11 Apply the distributive property to produce an equivalent expression: $x(y+z)$.

The distributive property, first introduced in Problem 4.9, allows you to multiply a quantity contained within grouping symbols—usually parentheses or brackets—by a single term. In this expression, the terms y and z within parentheses are each multiplied by x. Rewrite the expression, literally distributing the term outside the grouping symbols, multiplying it by each of the terms within.

$$\begin{aligned} x(y+z) &= x \cdot y + x \cdot z \\ &= xy + xz \end{aligned}$$

16.12 Apply the distributive property: $3(x+5)$.

Multiply the constant 3 by each of the terms within parentheses.

$$\begin{aligned} 3(x+5) &= 3 \cdot x + 3 \cdot 5 \\ &= 3x + 15 \end{aligned}$$

16.13 Apply the distributive property: $2(y+9z)$.

Distribute 2 to each term in the quantity. It is often helpful to indicate multiplication using parentheses (rather than dots), especially when terms have multiple factors or are negative.

$$\begin{aligned} 2(y+9z) &= 2(y) + 2(9z) \\ &= 2y + 18z \end{aligned}$$

In this context, the word "quantity" means "things in parentheses." You'd read $2(y + 9z)$ as, "Two times the quantity y plus $9z$."

16.14 Apply the distributive property: $-7(4x-5y-1)$.

This expression is the product of -7 and a quantity within parentheses. The quantity, unlike the preceding examples in this section, contains three terms, but the procedure described in Problems 16.11–16.13 applies; -7 is distributed to—and hence multiplied by—each term in the quantity.

$$-7(4x-5y-1)=-7(4x)-7(-5y)-7(-1)$$

Note that you are multiplying by a negative number (-7), so the addition signs that appeared between terms in the preceding examples are replaced by negative signs. Two of those signs change in the next step, because the second and third terms are the products of two negative numbers. Recall that multiplying two negative values produces a positive result.

$$=-28x+35y+7$$

This expression does not contain any like terms, so you cannot simplify it.

16.15 Apply the distributive property: $(x+10)x$.

In this expression, the term to be distributed appears on the right side of the parentheses. You may choose to rewrite the expression as $x(x+10)$, so it better resembles the preceding examples in this section, but that step is not necessary. Regardless of the order in which the expression is written, when a single value is multiplied by a quantity within grouping symbols, you can distribute that value through the grouping symbols, multiplying by each of the terms within.

$$\begin{aligned}(x+10)x &= x(x+10)\\ &= x(x)+x(10)\\ &= x^2+10x\end{aligned}$$

16.16 Apply the distributive property: $5y[8x-2]$.

The grouping symbols in this expression are brackets, but the difference between brackets and parentheses in an expression is purely cosmetic. One type of grouping symbol is not prioritized above another. Mathematically speaking, parenthesis and bracket notation are interchangeable.

$$\begin{aligned}5y[8x-2] &= 5y[8x]+5y[-2]\\ &= 40xy-10y\end{aligned}$$

16.17 Apply the distributive property: $-4z^3(5x^2 - 3y + 6z)$.

Distribute $-4z^3$ to each of the terms bounded by parentheses. Note that $-4z^3$ and $6z$ contain the same variable, so their product will contain z to the sum of the powers of z: $-4z^3(6z) = -24z^4$.

$$\begin{aligned}-4z^3\left(5x^2-3y+6z\right) &= -4z^3\left(5x^2\right)-4z^3\left(-3y\right)-4z^3\left(6z\right)\\ &= -20x^2z^3+12yz^3-24z^4\end{aligned}$$

16.18 Apply the distributive property: $6xy(-2x - 9y + 7)$.

Distributing the term $6xy$ to the first term within the quantity produces a term containing x^2; distributing to the second term in the quantity produces a term containing y^2.

$$\begin{aligned}6xy\left(-2x-9y+7\right) &= 6xy\left(-2x\right)+6xy\left(-9y\right)+6xy(7)\\ &= -12x^2y-54xy^2+42xy\end{aligned}$$

16.19 Apply the distributive property: $4x^2y^3\left(5x^3 - 8x^2y + xy^2 - 3y^3\right)$.

Add the powers of like variables when you distribute $4x^2y^3$ to each term in parentheses.

$$\begin{aligned}4x^2y^3\left(5x^3-8x^2y+xy^2-3y^3\right) &= 4x^2y^3\left(5x^3\right)+4x^2y^3\left(-8x^2y\right)+4x^2y^3\left(xy^2\right)+4x^2y^3\left(-3y^3\right)\\ &= 20x^{2+3}y^3-32x^{2+2}y^{3+1}+4x^{2+1}y^{3+2}-12x^2y^{3+3}\\ &= 20x^5y^3-32x^4y^4+4x^3y^5-12x^2y^6\end{aligned}$$

16.20 Apply the distributive property and simplify:

$$-9x^2(10x - 3) + 6x(x^2 - 2x) - 7(x + 4).$$

The distributive property is applied three times in this expression. Expand the expressions, multiplying the quantities grouped by parentheses by the outer, preceding factors.

Multiply the stuff inside each set of parentheses by the term that's stuck to the outside of that particular set of parentheses.

$$\begin{aligned}&-9x^2\left(10x-3\right)+6x\left(x^2-2x\right)-7\left(x+4\right)\\ &\quad= -9x^2\left(10x\right)-9x^2\left(-3\right)+6x\left(x^2\right)+6x\left(-2x\right)-7\left(x\right)-7\left(4\right)\\ &\quad= -90x^{2+1}+27x^2+6x^{1+2}-12x^{1+1}-7x-28\\ &\quad= -90x^3+27x^2+6x^3-12x^2-7x-28\end{aligned}$$

Rewrite the expression, grouping like terms.

$$\begin{aligned}&= \left(-90x^3+6x^3\right)+\left(27x^2-12x^2\right)-7x-28\\ &= -84x^3+15x^2-7x-28\end{aligned}$$

The FOIL Method

First, Outer, Inner, Last

16.21 Demonstrate and explain the FOIL method for multiplying a pair of expressions containing two terms each, using $(x+y)(a+b)$ as an example.

The fancy word for a variable expression that contains two terms is "binomial," so you can say that FOIL is used to multiply pairs of binomials.

Problems 16.11–16.20 explore the distributive property, which allows you to multiply an expression within grouping symbols by a single term. The FOIL technique also applies to a specific kind of multiplication problem; it calculates the product of two quantities, each containing exactly two terms.

To calculate the product $(x+y)(a+b)$, you must first multiply four pairs of terms and then add the results. The letters in FOIL serve as a mnemonic device to identify the pairs you are required to multiply: First, Outer, Inner, and Last.

- **First:** The first term in each of the quantities, literally the term that appears first—the furthest to the left—in each set of parentheses. In this expression, the first term of $(x+y)$ is x; the first term of $(a+b)$ is a. Therefore, the product of the first terms is $x \cdot a$.
- **Outer:** The outer terms of the product are the terms that appear at the far left and far right side of the expression. In this example, the outer terms are x and b, as the terms y and a are physically sandwiched between x on the left and b on the right. Thus, the product of the outer terms is $x \cdot b$.
- **Inner:** As explained above, y and a are the inner terms of this product because they are located between the outer terms x and b. Thus, the product of the inner terms is $y \cdot a$.
- **Last:** The last terms appear at the end of each set of parentheses—the right-hand terms in each group. In this expression, the last terms are y and b; their product is $y \cdot b$.

Each of the letters in FOIL represents a product that is calculated in the bullets above. The product $(x+y)(a+b)$ is equal to the sum of those four individual products.

$$\begin{aligned}(x+y)(a+b) &= (x \cdot a)+(x \cdot b)+(y \cdot a)+(y \cdot b)\\ &= ax+bx+ay+by\end{aligned}$$

Remember to list the variables within a term in alphabetical order. For example, according to mathematics convention, $x \cdot a$ is best expressed ax.

16.22 Write the product as a single, simplified expression: $(x+2)(x+7)$.

Identify the terms defined by the FOIL method.

	$(x+2)$	$(x+7)$
First	x	x
Outer	x	7
Inner	2	x
Last	2	7

Calculate the products in each row of the table above and add the results.

$$\begin{aligned}(x+2)(x+7) &= (x\cdot x)+(x\cdot 7)+(2\cdot x)+(2\cdot 7)\\ &= x^2+7x+2x+14\end{aligned}$$

Finish simplifying the expression by combining like terms.

$$= x^2+9x+14$$

16.23 Write the product as a single, simplified expression: $(x+9)(x-9)$.

Construct a chart, as demonstrated in Problem 16.22, which identifies the pairs of terms required to apply the FOIL method.

	$(x+9)$	$(x-9)$
First	x	x
Outer	x	-9
Inner	9	x
Last	9	-9

Multiply the terms in each row and calculate the sum.

$$\begin{aligned}(x+9)(x-9) &= (x)(x)+(x)(-9)+(9)(x)+(9)(-9)\\ &= x^2-9x+9x-81\end{aligned}$$

The like terms in the expression, $-9x$ and $9x$, are opposites, so the final answer will only contain two terms.

When you multiply $(x + \text{number})$ by $(x - \text{number})$, and "number" represents the same value in both groups, the product is $x^2 - (\text{number})^2$. In this case, $(x+9)(x-9) = x^2 - 9^2$.

$$\begin{aligned}&= x^2+0x-81\\ &= x^2-81\end{aligned}$$

16.24 Write the product as a single, simplified expression: $(y-3)(2y+1)$.

Identify the terms required by the FOIL method.

	$(y-3)$	$(2y+1)$
First	y	$2y$
Outer	y	1
Inner	-3	$2y$
Last	-3	1

Apply the FOIL method to calculate the product.

$$\begin{aligned}(y-3)(2y+1) &= (y)(2y)+(y)(1)+(-3)(2y)+(-3)(1)\\ &= 2y^2+y-6y-3\\ &= 2y^2+(y-6y)-3\\ &= 2y^2-5y-3\end{aligned}$$

16.25 Write the product as a single, simplified expression: $(4x-5)(3x-8)$.

You may wish to use the chart presented in Problems 16.22–16.24 to organize the terms into first, outer, inner, and last pairs. However, it is not necessary to do so. In the solution below (and in subsequent solutions) the chart is omitted.

$$\begin{aligned}(4x-5)(3x-8) &= (4x)(3x)+(4x)(-8)+(-5)(3x)+(-5)(-8)\\ &= 12x^2-32x-15x+40\\ &= 12x^2-47x+40\end{aligned}$$

16.26 Write the product as a single, simplified expression: $(2x^2+7)(7x^2-1)$.

Apply the FOIL method to calculate the product. Note that the first terms, as defined by the FOIL method, are $2x^2$ and $7x^2$; they have a product of $14x^{2+2}=14x^4$.

$$\begin{aligned}(2x^2+7)(7x^2-1) &= (2x^2)(7x^2)+(2x^2)(-1)+(7)(7x^2)+(7)(-1)\\ &= 14x^4-2x^2+49x^2-7\\ &= 14x^4+47x^2-7\end{aligned}$$

16.27 Expand and simplify the expression: $(4y+3)^2$.

To square a quantity, multiply that quantity times itself. In the same way you can rewrite a^2 as $a \cdot a$, you can express $(4y+3)^2$ as the product $(4y+3)(4y+3)$. Apply the FOIL method.

$$\begin{aligned}(4y+3)^2 &= (4y+3)(4y+3)\\ &= (4y)(4y)+(4y)(3)+(3)(4y)+(3)(3)\\ &= 16y^2+12y+12y+9\\ &= 16y^2+24y+9\end{aligned}$$

More generally, squaring the variable expression $(a+b)$ produces the expression $a^2+2ab+b^2$. In the context of this problem, $a=4y$ and $b=3$, so $(a+b)^2$ is equal to $(4y)^2+2(4y)(3)+(3)^2$, or $16y^2+24y+9$.

16.28 Write the product as a single, simplified expression: $(x-4y)(8x-7y)$.

Apply the FOIL method. In this expression, the first terms are x and $8x$, the outer terms are x and $-7y$, the inner terms are $-4y$ and $8x$, and the last terms are $-4y$ and $-7y$.

$$\begin{aligned}(x-4y)(8x-7y) &= (x)(8x)+(x)(-7y)+(-4y)(8x)+(-4y)(-7y)\\ &= 8x^2-7xy-32xy+28y^2\end{aligned}$$

Combine like terms $-7xy$ and $-32xy$.

$$= 8x^2-39xy+28y^2$$

16.29 Write the product as a single, simplified expression: $(3x^2+x)(10x-9)$.

Apply the FOIL method to calculate the product.

$$\begin{aligned}(3x^2+x)(10x-9) &= (3x^2)(10x)+(3x^2)(-9)+(x)(10x)+(x)(-9)\\ &= 30x^3-27x^2+10x^2-9x\end{aligned}$$

Simplify the expression by combining like terms $-27x^2$ and $10x^2$.

$$= 30x^3-17x^2-9x$$

16.30 Demonstrate that $(x+y)^2 \neq x^2+y^2$.

This is a very common error. Students see $(x+y)^2$ and think they should square the things in the parentheses one at a time. Unfortunately, that only works if there's a single term in parentheses. In other words, $(2xy)^3 = 2^3 \cdot x^3 \cdot y^3$, but $(2+x+y)^3 \neq 2^3+x^3+y^3$.

Rewrite $(x+y)^2$ as $(x+y)(x+y)$ and apply the FOIL method.

$$\begin{aligned}(x+y)^2 &= (x+y)(x+y)\\ &= (x\cdot x)+(x\cdot y)+(y\cdot x)+(y\cdot y)\\ &= x^2+xy+xy+y^2\\ &= x^2+2xy+y^2\end{aligned}$$

Thus, $(x+y)^2 = x^2+2xy+y^2$. The product does not equal x^2+y^2.

Multiplication

Multiply coefficients, add exponents

16.31 Explain how to multiply variable expressions when the FOIL method does not apply, using $(a+b)(x+y+z)$ as an example.

As Problem 16.21 explains, the FOIL method may be used to multiply two quantites only if each contains two terms. In this problem, the quantity $(x+y+z)$ contains three terms, so you cannot apply the FOIL method—the terms "first," "outer," "inner," and "last" are not defined for expressions containing more than two terms.

In order to multiply the expressions, you will distribute each of the terms in the left quantity $(a+b)$ individually to each of the terms in the right quantity $(x+y+z)$.

Begin with the first term in the left quantity (a); distribute it to each of the terms in the right quantity.

$$a(x+y+z) = ax + ay + az$$

Now distribute the second term of the left quantity (b) to each term of the right quantity.

$$b(x+y+z) = bx + by + bz$$

The final product is equal to the sum of the six individual products identified above.

$$(a+b)(x+y+z) = ax + ay + az + bx + by + bz$$

Each term in the left quantity, one at a time, gets multiplied by the entire right quantity: $(a+b)(x+y+z) = a(x+y+z) + b(x+y+z)$

Note: In Problems 16.32–16.33, you demonstrate the commutative property of multiplication. The order in which the product of expressions is written does not affect the answer.

16.32 Write the product as a single, simplified expression: $(x+2)(x^2+4x+5)$.

Apply the procedure described in Problem 16.31. Begin by distributing the first term of the left quantity (x) to each term of the right quantity (x^2+4x+5).

$$x(x^2+4x+5) = x^3 + 4x^2 + 5x$$

Now distribute the second term of the left quantity (2) to each term of the right quantity.

$$2(x^2+4x+5) = 2x^2 + 8x + 10$$

The product $(x+2)(x^2+4x+5)$ is equal to the sum of the individual products identified above. Combine like terms to simplify the expression.

$$\begin{aligned}(x+2)(x^2+4x+5) &= (x^3+4x^2+5x)+(2x^2+8x+10)\\ &= x^3+(4x^2+2x^2)+(5x+8x)+10\\ &= x^3+6x^2+13x+10\end{aligned}$$

Note: In Problems 16.32–16.33, you demonstrate the commutative property of multiplication. The order in which the product of expressions is written does not affect the answer.

16.33 Verify your solution to Problem 16.32 by writing the product as a single, simplified expression: $(x^2+4x+5)(x+2)$.

Whereas Problem 16.32 has two terms in the left expression, this permutation of the same product has three terms in the left expression. The procedure remains the same—distribute x^2, $4x$, and 5—one at a time—to the terms in the right expression.

$$\begin{aligned}x^2(x+2) &= x^3+2x^2\\ 4x(x+2) &= 4x^2+8x\\ 5(x+2) &= 5x+10\end{aligned}$$

Add the individual products and combine like terms.

$$\begin{aligned}(x^2+4x+5)(x+2) &= (x^3+2x^2)+(4x^2+8x)+(5x+10)\\ &= x^3+(2x^2+4x^2)+(8x+5x)+10\\ &= x^3+6x^2+13x+10\end{aligned}$$

The product exactly matches, and therefore verifies, the solution to Problem 16.32.

16.34 Write the product as a single, simplified expression: $(x-3)(8x^2+2x-9)$.

Multiply each term of the left quantity by the entire right quantity and add the individual products.

$$(x-3)(8x^2+2x-9) = x(8x^2+2x-9) - 3(8x^2+2x-9)$$

Starting with this problem, the book writes all of the distribution as a single step, instead of treating each term in the left quantity as an entirely separate step and then adding everything together at the end. No change in the process—just a different way to write it.

Notice that the second term of the left quantity (–3) is negative. Make sure to include the sign when distributing.

$$\begin{aligned}&= (8x^3+2x^2-9x)+(-24x^2-6x+27)\\ &= 8x^3+(2x^2-24x^2)+(-9x-6x)+27\\ &= 8x^3-22x^2-15x+27\end{aligned}$$

16.35 Write the product as a single, simplified expression: $(3y-4)(-5y^3+11y^2-y+6)$.

The right quantity in this product contains four terms. This does not affect the technique you use to multiply the expressions; each time you distribute the terms in the left quantity ($3y$ and -4), it will produce four products. Combine like terms in the eight resulting products to simplify.

$$\begin{aligned}(3y-4)(-5y^3+11y^2-y+6)\\ &= 3y(-5y^3+11y^2-y+6)-4(-5y^3+11y^2-y+6)\\ &= (-15y^4+33y^3-3y^2+18y)+(20y^3-44y^2+4y-24)\\ &= -15y^4+(33y^3+20y^3)+(-3y^2-44y^2)+(18y+4y)-24\\ &= -15y^4+53y^3-47y^2+22y-24\end{aligned}$$

16.36 Verify that the technique described in Problems 16.31–16.35 produces the same result as the FOIL method when multiplying a pair of expressions containing two terms. Use $(w+8)(10w-3)$ as an example.

Apply the technique described in Problems 16.31–16.35, multiplying $(10w-3)$ first by w and then by 8.

$$\begin{aligned}(w+8)(10w-3) &= w(10w-3)+8(10w-3)\\ &= 10w^2-3w+80w-24\end{aligned}$$

Before you simplify like terms, notice that this result is exactly the same expression that is produced using the FOIL method.

$$\begin{aligned}(w+8)(10w-3) &= (w)(10w)+(w)(-3)+(8)(10w)+(8)(-3)\\ &= 10w^2-3w+80w-24\end{aligned}$$

In fact, not only are the expressions equivalent, they are exactly alike—the terms even appear in the same order. The FOIL method is nothing more than the general multiplication algorithm with an acronymic name. In either case, $(w+8)(10w-3)=10w^2+77w-24$.

16.37 Write the product as a single, simplified expression: $(y+1)(y+2)(y+3)$.

According to the order of operations, multiplication should be completed from left to right. Begin by multiplying $(y+1)$ and $(y+2)$.

$$\begin{aligned}(y+1)(y+2)(y+3) &= [(y+1)(y+2)](y+3)\\ &= [y^2+2y+y+2](y+3)\\ &= [y^2+3y+2](y+3)\end{aligned}$$

You can reverse the quantities if you want to: $(y+3)(y^2+3y+2)$.

Distribute each of the terms in the left quantity to the entire right quantity and combine like terms.

$$\begin{aligned}&= y^2(y+3)+3y(y+3)+2(y+3)\\&= y^3+3y^2+3y^2+9y+2y+6\\&= y^3+6y^2+11y+6\end{aligned}$$

16.38 Write the product as a single, simplified expression: $(4w+1)(2w-6)(w-9)$.

Like Problem 16.37, this problem is the product of three quantities, each containing two terms. Begin by calculating $(4w+1)(2w-6)$.

$$\begin{aligned}(4w+1)(2w-6)(w-9)&=[(4w+1)(2w-6)](w-9)\\&=[8w^2-24w+2w-6](w-9)\\&=[8w^2-22w-6](w-9)\end{aligned}$$

Multiply the product calculated above $(8w^2-22w-6)$ by $(w-9)$, the remaining quantity from the original expression.

$$\begin{aligned}&=8w^2(w-9)-22w(w-9)-6(w-9)\\&=8w^3-72w^2-22w^2+198w-6w+54\\&=8w^3-94w^2+192w+54\end{aligned}$$

16.39 Write the product as a single, simplified expression: $(x^2+5x+3)^2$.

Expand the expression, writing the squared quantity as the product of the quantity multiplied times itself.

$$(x^2+5x+3)^2=(x^2+5x+3)(x^2+5x+3)$$

Distribute each term of the left quantity to the entire right quantity.

$$\begin{aligned}&=x^2(x^2+5x+3)+5x(x^2+5x+3)+3(x^2+5x+3)\\&=(x^4+5x^3+3x^2)+(5x^3+25x^2+15x)+(3x^2+15x+9)\end{aligned}$$

Group like terms and simplify the expression.

$$\begin{aligned}&=x^4+(5x^3+5x^3)+(3x^2+25x^2+3x^2)+(15x+15x)+9\\&=x^4+10x^3+31x^2+30x+9\end{aligned}$$

16.40 Write the product as a single, simplified expression: $(2z-1)^2(2z+1)$.

Expand the expression and multiply the first two quantities, as you read from left to right.

$$\begin{aligned}(2z-1)^2(2z+1) &= (2z-1)(2z-1)(2z+1)\\ &= \left[(2z-1)(2z-1)\right](2z+1)\\ &= \left[4z^2-2z-2z+1\right](2z+1)\\ &= \left[4z^2-4z+1\right](2z+1)\end{aligned}$$

Multiply the product calculated above $(4z^2-4z+1)$ by the rightmost quantity in the original expression $(2z+1)$.

$$\begin{aligned}&= 4z^2(2z+1)-4z(2z+1)+1(2z+1)\\ &= 8z^3+4z^2-8z^2-4z+2z+1\\ &= 8z^3-4z^2-2z+1\end{aligned}$$

16.41 Write the product as a single, simplified expression:

$$(x^2+8x-4)^2+(9x-5)(x^2+2x-7)$$

Rather than simplify the entire expression at once, calculate the two products in the expression independently and then calculate the sum. Begin by expanding and simplifying $(x^2+8x-4)^2$.

$$\begin{aligned}(x^2+8x-4)^2 &= (x^2+8x-4)(x^2+8x-4)\\ &= x^2(x^2+8x-4)+8x(x^2+8x-4)-4(x^2+8x-4)\\ &= (x^4+8x^3-4x^2)+(8x^3+64x^2-32x)+(-4x^2-32x+16)\\ &= x^4+(8x^3+8x^3)+(-4x^2+64x^2-4x^2)+(-32x-32x)+16\\ &= x^4+16x^3+56x^2-64x+16\end{aligned}$$

Now calculate the product $(9x-5)(x^2+2x-7)$.

$$\begin{aligned}(9x-5)(x^2+2x-7) &= 9x(x^2+2x-7)-5(x^2+2x-7)\\ &= 9x^3+18x^2-63x-5x^2-10x+35\\ &= 9x^3+(18x^2-5x^2)+(-63x-10x)+35\\ &= 9x^3+13x^2-73x+35\end{aligned}$$

Add both of the results above and combine like terms to simplify.

$$\begin{aligned}&(x^2+8x-4)^2+(9x-5)(x^2+2x-7)\\ &\quad= \left[x^4+16x^3+56x^2-64x+16\right]+\left[9x^3+13x^2-73x+35\right]\\ &\quad= x^4+(16x^3+9x^3)+(56x^2+13x^2)+(-64x-73x)+(16+35)\\ &\quad= x^4+25x^3+69x^2-137x+51\end{aligned}$$

Chapter 17

BASIC LINEAR EQUATIONS

What value of x makes the statement true?

While most of the preceding chapters have addressed expressions—strings of mathematical terms, factors, and operations that do not contain an equal sign—this chapter begins the study of equations in earnest. Because an equation is little more than a pair of equal expressions, the concepts and skills explored thus far in the book will still prove useful and, in fact, necessary.

The chapter begins by identifying the unique skills that are required to solve equations. It then graduates from simple, one-step equations to those requiring multiple steps to solve. It concludes with a review of proportions, providing the rationale for the techniques first discussed in the context of percentages in Chapter 12.

Solving equations is much more fun, at least in my humble opinion, than long division or adding fractions. An equation is a mystery, a game. It's a more advanced version of "Try to guess what number I am thinking of." For example, solving the equation $2x + 7 = 29$ is the same as answering the question, "What number, when multiplied by two and added to seven, is equal to twenty-nine?"

I'll admit that it's perhaps not the most fun game ever created, but at least you know if your answer is right when you're done. After all, if multiplying your answer by 2 and adding 7 doesn't give you 29, then you know to go back and try again. In this chapter, you'll figure out how to come up with that mystery number ($x = 11$ is the solution to $2x + 7 = 29$, in case you're wondering).

Equation Solving Techniques

Keep the equation balanced

A linear equation is an equation that contains only one kind of variable (usually x) that is raised to the first power. Equations that contain x^2, for example, are called quadratic equations, and they are solved in a completely different way.

17.1 Explain what it means to solve for, or isolate, a variable in a linear equation.

When you solve for the variable in a linear equation, you are physically isolating it on one side of the equation. No matter what the original equation looks like, you perform a series of well-defined steps, manipulating the appearance of the equation without affecting its fundamental defining quality: the expressions on both sides of the equal sign have the same value. The eventual goal of this process is a single variable, alone on one side of the equal sign. Thus, the solution to any linear equation is "$x =$" or "$= x$."

17.2 Solve the equation and identify the property of equality that allows you to isolate the variable: $x + 3 = 11$.

The solution to the equation $x + 3 = 11$ answers the question, "What number, if added to three, is equal to eleven?" Although you can make an educated guess and eventually discern the answer using trial and error, it is important to understand the properties of equality that allow you to solve equations. As you proceed through the chapter, the equations will become increasingly difficult, and intuition alone will be insufficient to find solutions.

As Problem 17.1 explains, when you are solving linear equations, your goal is to isolate x on one side of the equal sign, usually the left side for reasons of convention although this has no impact on the answer. Notice that x already appears on the left side of the equation; however, the term 3 also appears left of the equal sign.

In order to isolate x, you must eliminate 3 from the left side of the equation, but simply subtracting 3 will invalidate the statement. It is very important to preserve an equality statement, or the solutions that result will also be invalid.

Happily, it is a trivial matter to maintain equality in a simple linear equation. According to the subtraction property of equality, you may subtract any real number value from one side of an equation as long as you subtract the same value from the other side as well.

Think of the two sides of the equation as perfectly balanced weights at the ends of a playground seesaw. If you subtract three pounds from one side, the only way to keep things balanced is to subtract three pounds from the other side, too.

To summarize, you need to subtract 3 from the left side of the equation, and in order to do so, you must subtract 3 from the right side as well, as demonstrated below.

$$\begin{array}{rcrcr} x & + & 3 & = & 11 \\ \hline & - & 3 & & -3 \end{array}$$

Combine like terms, which are written above and beneath one another in the same columns.

$$\begin{array}{ccccc} x & + & 3 & = & 11 \\ & - & 3 & & -3 \\ \hline x & + & 0 & = & 8 \end{array}$$

Omit "+ 0" from the left side of the equation.

$$x = 8$$

You conclude that $x = 8$ is the number that, when added to 3, is equal to 11.

17.3 Solve the equation and identify the property of equality that allows you to isolate the variable: $x - 5 = -2$.

Like Problem 17.2, this problem contains a variable (x) and a constant (-5) on the left side of the equal sign. In order to solve the equation, you must isolate x; this requires you to eliminate -5 from the left side of the equation. Eliminate the constant using its opposite, in this case $+5$. Add 5 to both sides of the equation, as demonstrated below.

$$\begin{array}{ccccc} x & - & 5 & = & -2 \\ & + & 5 & & +5 \\ \hline x & + & 0 & = & 3 \end{array}$$

The addition property of equality allows you to add any real number value to one side of an equation, as long as that value is also added to the other side of the equation as well. In this problem, adding 5 to both sides of the equation isolates x, and the solution is $x = 3$.

17.4 Verify your solution to Problem 17.3.

According to Problem 17.3, $x = 3$ is the solution to the equation $x - 5 = -2$. To verify the solution, replace x in the original equation with the proposed solution, 3. Place parentheses around the real number value when you substitute it into the equation to avoid errors involving signs. In this problem, the parentheses are not particularly helpful, but it is a good practice to employ nonetheless.

$$\begin{aligned} x - 5 &= -2 \\ (3) - 5 &= -2 \\ 3 - 5 &= -2 \\ -2 &= -2 \end{aligned}$$

Because you end with a true statement ($-2 = -2$), you conclude that $x = 3$ is the solution to the equation $x - 5 = -2$.

17.5 Solve the equation and identify the property of equality that allows you to isolate the variable: $9x = 54$.

Consider the operations applied thus far to isolate a variable:

- In Problem 17.2, the left side of the equation is $x + 3$. To eliminate 3, you combine it with its opposite, -3. The opposite of addition is subtraction, so you apply the subtraction property of equality.
- In Problem 17.3, the left side of the equation is $x - 5$. To eliminate -5, you combine it with its opposite, $+5$. The opposite of subtraction is addition, so you apply the addition property of equality.

You can divide both sides by any real number except zero. Division by zero is not allowed.

In this problem, x is multiplied by the coefficient 9. The opposite of multiplication is division, so to eliminate 9 from the left side of the equation you must divide both sides of the equation by 9. This is permitted, according to the division property of equality.

$$9x = 54$$

$$\frac{9x}{9} = \frac{54}{9}$$

Note that $9 \div 9 = 1$ and $54 \div 9 = 6$.

$$1x = 6$$

You do not need to write a coefficient of 1—that coefficient is assumed if none is written.

$$x = 6$$

17.6 Solve the equation and identify the property of equality that allows you to isolate the variable: $\frac{2}{3}x = 26$.

According to Problem 17.5, in order to eliminate whole number coefficients, you should divide both sides of the equation by the coefficient. In this problem, the coefficient of x is a fraction. To eliminate fractional coefficients, apply the multiplication property of equality, which allows you to multiply both sides of the equation by the reciprocal of the fraction.

Technically, you're dividing both sides of the equation by $\frac{2}{3}$ because dividing by a fraction is the same thing as multiplying by its reciprocal. (Flip back to Problem 9.15 for proof!)

The coefficient of x in this equation is $\frac{2}{3}$, so multiply both sides of the equation by $\frac{3}{2}$.

$$\frac{3}{2}\left(\frac{2}{3}\right)x = \frac{3}{2}(26)$$

When you multiply the fractions on the left side of the equation, the result is $\frac{6}{6}$, which equals 1.

$$\frac{3\cdot 2}{2\cdot 3}x=\frac{3}{2}(26)$$
$$\frac{6}{6}x=\frac{3}{2}(26)$$
$$1x=\frac{3}{2}(26)$$
$$x=\frac{3}{2}(26)$$

To simplify the right side of the equation, divide 26 by 1 to express it as a fraction. Notice that numerator value 26 and denominator value 2 have greatest common factor 2, so reduce the fractions before multiplying.

$$\begin{aligned} x &= \frac{3}{2}\left(\frac{26}{1}\right) \\ &= \frac{3}{1\,\cancel{2}}\left(\frac{13\,\cancel{26}}{1}\right) \\ &= \frac{3}{1}\left(\frac{13}{1}\right) \\ &= 39 \end{aligned}$$

17.7 Verify your solution to Problem 17.6.

According to Problem 17.6, the solution to the equation $\frac{2}{3}x=26$ is $x=39$. Replace x in the original equation with the solution, 39.

$$\frac{2}{3}x=26$$
$$\frac{2}{3}(39)=26$$
$$\frac{2}{3}\left(\frac{39}{1}\right)=26$$

Numerator value 39 and denominator value 3 share greatest common factor 3; use it to reduce the fractions and then multiply.

$$\frac{2}{1\,\cancel{3}}\left(\frac{13\,\cancel{39}}{1}\right)=26$$
$$\frac{2}{1}\left(\frac{13}{1}\right)=26$$
$$\frac{26}{1}=26$$
$$26=26$$

The final statement (and thus each of the preceding statements in the problem) is true, so you are assured that $x=39$ is the solution to the equation.

And it's the only one—linear equations only have one solution

Basic Linear Equations

Only two steps required

17.8 List the steps required to solve a linear equation; assume the equation is written in terms of x.

Problems 17.1–17.7 underscore the importance of isolating a variable in order to solve an equation. By ensuring that the variable appears by itself—with a coefficient of 1—on one side of the equation, you are assured that the value appearing on the other side of the equation is the solution to the linear equation. However, isolating a variable is rarely as simple as adding, subtracting, multiplying, or dividing by a single real number.

The following steps outline the general procedure you should follow to solve a linear equation. Remember, the final goal of these steps is to isolate the variable on one side of the equation.

1. Simplify the expressions on both sides of the equal sign independently.
2. Use the addition and subtraction properties of equality to move the x-terms to one side of the equal sign.
3. Use the addition and subtraction properties of equality to move all other terms to the opposite side of the equal sign, as demonstrated in Problems 17.2–17.3.
4. Combine like terms so that a single x-term appears on one side of the equal sign and a single constant appears on the other side.
5. Eliminate the coefficient of the x-term using the multiplication property of equality or the division property of equality, as demonstrated in Problems 17.5–17.6.

Not all equations will require you to complete each of the steps, but any linear equation can be solved by completing these steps in the prescribed order.

17.9 Apply the steps listed in Problem 17.8 to solve the equation: $2x - 7 = 19$.

Consider the steps outlined in Problem 17.8. Each is listed below and is applied to the equation $2x - 7 = 19$.

1. *Simplify the expressions on both sides of the equal sign independently.* The left side of the equation contains unlike terms $2x$ and -7, which cannot be combined. The right side contains the constant 19. This step does not apply to this equation, as neither side can be simplified.
2. *Move all of the x-terms to one side of the equation.* Only a single x-term appears in the equation: $2x$. Thus, all of the x-terms are already located on a single side of the equation.

3. *Move all of the other terms to the other side of the equation.* In Step 2, the left side of the equation is designated as the side that will contain the variable term $2x$. Hence, the right side should contain the constant. To eliminate –7 from the left side of the equation, add 7 to both sides.

$$\begin{array}{rcrcr} 2x & - & 7 & = & 19 \\ & + & 7 & & +7 \\ \hline 2x & + & 0 & = & 26 \end{array}$$

The constants –7 and +7 on the left side of the equation are also combined. They add up to 0.

4. *Combine like terms on both sides of the equal sign.* Technically, this was completed in Step 3; constants 19 and 7 are combined to produce a sum of 26.

$$2x = 26$$

5. *Eliminate the coefficient of the x-term.* To complete the process of isolating x on the left side of the equation, divide both sides of the equation by its coefficient, 2.

$$\frac{2x}{2} = \frac{26}{2}$$
$$x = 13$$

The solution to the equation $2x - 7 = 19$ is $x = 13$.

17.10 Solve the equation and verify your solution: $4y - 11 = 1$.

The steps outlined in Problem 17.9 are also applied in this problem, but the detailed rationale for each step is omitted. Begin by adding 11 to both sides of the equation, in order to move all of the constant terms to the right side of the equation.

$$\begin{array}{rcrcr} 4y & - & 11 & = & 1 \\ & + & 11 & & +11 \\ \hline 4y & + & 0 & = & 12 \end{array}$$

Divide both sides of the equation $4y = 12$ by the coefficient of y.

$$4y = 12$$
$$\frac{4y}{4} = \frac{12}{4}$$
$$y = 3$$

To verify your solution $(y = 3)$, substitute it into the original equation.

$$\begin{aligned} 4y - 11 &= 1 \\ 4(3) - 11 &= 1 \\ 12 - 11 &= 1 \\ 1 &= 1 \end{aligned}$$

The statement $1 = 1$ is true, so the solution $y = 3$ is correct.

17.11 Solve the equation: $3x + 5 = 0$.

Separate the variable term $3x$ and constant term 5 on opposite sides of the equal sign by subtracting 5 from both sides of the equation.

$$\begin{array}{rcrcr} 3x & + & 5 & = & 0 \\ & - & 5 & & -5 \\ \hline 3x & + & 0 & = & -5 \end{array}$$

Divide both sides of the equation $3x = -5$ by 3.

$$\frac{3x}{3} = \frac{-5}{3}$$
$$x = -\frac{5}{3}$$

Do not convert the improper fraction into a mixed number or decimal unless specifically instructed to do so.

17.12 Solve the equation: $8w = 9w + 6$.

Both sides of the equation contain variable terms. The second step of the linear equation solving process, as explained in Problem 17.8, is to apply the addition or subtraction property of equality to move all of the variable terms to one side of the equal sign. The right side of this equation contains a constant, so designate the left side of the equation for the variable terms.

Subtract $9w$ from both sides of the equation to eliminate the variable term from the right side of the equation.

$$\begin{array}{rcrcr} 8w & = & 9w & + & 6 \\ -9w & & -9w & & \\ \hline -1w & = & 0 & + & 6 \end{array}$$

The equation $-1w = 6$ may also be written $-w = 6$. It is not solved for w, because its coefficient is not +1. In order to eliminate the -1 coefficient, either multiply or divide both sides of the equation by -1; both operations have the same effect.

$$-w = 6$$
$$(-1)(-w) = (-1)6$$
$$w = -6$$

This line says, "The opposite of w is 6." That also means the opposite of 6 is equal to w. In other words, $w = -6$.

Note: Problems 17.13–17.14 refer to the equation $-10-\frac{2}{3}y=5$.

17.13 Solve the equation for y.

Eliminate the constant term –10 from the left side of the equation by adding 10 to both sides.

$$\begin{array}{rcrcr} -10 & - & \frac{2}{3}y & = & 5 \\ +10 & & & & +10 \\ \hline & - & \frac{2}{3}y & = & 15 \end{array}$$

To solve for y, you must eliminate the fractional coefficient. According to Problem 17.6, you need to multiply both sides of the equation by the reciprocal of $-\frac{2}{3}$, which is $-\frac{3}{2}$.

$$\left(-\frac{3}{2}\right)\left(-\frac{2}{3}y\right)=\left(-\frac{3}{2}\right)(15)$$
$$\left(-\frac{3}{2}\right)\left(-\frac{2}{3}y\right)=\left(-\frac{3}{2}\right)\left(\frac{15}{1}\right)$$
$$\frac{6}{6}y=-\frac{45}{2}$$
$$y=-\frac{45}{2}$$

Note: Problems 17.13–17.14 refer to the equation $-10-\frac{2}{3}y=5$.

17.14 Verify your solution to Problem 17.13.

According to Problem 17.13, $y=-\frac{45}{2}$. Substitute this value for y in the original equation.

$$-10-\frac{2}{3}y=5$$
$$-10-\frac{2}{3}\left(-\frac{45}{2}\right)=5$$

Reduce the product, dividing pairs of numerator and denominator values by their greatest common factors. Divide 2 and 2 by 2; divide 45 and 3 by 3.

$$-10-\frac{1\,\cancel{2}}{1\,\cancel{3}}\left(-\frac{15\,\cancel{45}}{1\,\cancel{2}}\right)=5$$
$$-10-\frac{1}{1}\left(-\frac{15}{1}\right)=5$$
$$-10-1(-15)=5$$

Simplify the left side of the equation; remember that multiplication precedes addition: $(-1)(-15) = +15$.

$$\begin{aligned} -10+15&=5 \\ 5&=5 \end{aligned}$$

Because the statement $5 = 5$ is true, $y=-\frac{45}{2}$ is the correct solution to the equation $-10-\frac{2}{3}y=5$.

Equations Requiring Multiple Steps

Slightly more complicated equations

17.15 Solve the equation: $3(x+2)=-18$.

Expand the left side of the equation by applying the distributive property.

$$\begin{aligned} 3(x)+3(2)&=-18 \\ 3x+6&=-18 \end{aligned}$$

Isolate the x-term on the left side of the equation by subtracting 6 from both sides.

$$\begin{array}{rcrcr} 3x & + & 6 & = & -18 \\ & - & 6 & & -6 \\ \hline 3x & & & = & -24 \end{array}$$

Divide both sides of the equation by 3 to solve for x.

$$\begin{aligned} \frac{3x}{3}&=\frac{-24}{3} \\ x&=-8 \end{aligned}$$

17.16 Solve the equation: $17-6x=5(x-1)$.

Apply the distributive property to expand the right side of the equation.

$$17-6x=5x-5$$

Add $6x$ to both sides of the equation.

This time the book is moving all of the x-terms to the right side of the equation instead of the left. Why? Probably just to show you that the x's don't always have to go on the left. If you subtract 17 from, and subtract 5x to, both sides, you get the same answer with the x's on the left.

$$\begin{array}{rcrcrcr} 17 & - & 6x & = & 5x & - & 5 \\ & + & 6x & = & +6x & & \\ \hline 17 & + & 0 & = & 11x & - & 5 \end{array}$$

Add 5 to both sides of the equation $17 = 11x - 5$ to isolate the x-term on the right side of the equation.

$$\begin{array}{rcrcr} 17 & = & 11x & - & 5 \\ +5 & & & + & 5 \\ \hline 22 & = & 11x & + & 0 \end{array}$$

Divide both sides of the equation by the coefficient of x to solve.

$$\frac{22}{11} = \frac{11x}{11}$$
$$2 = x$$

17.17 Solve the equation: $-2[3(x-4)+10] = 9x - 6$.

This equation contains two sets of grouping symbols, one nested within the other. Simplify the expression within the brackets by distributing 3 to the quantity $(x-4)$ and then add 10 to the result.

$$-2[3(x-4)+10] = 9x-6$$
$$-2[3(x)+3(-4)+10] = 9x-6$$
$$-2[3x-12+10] = 9x-6$$
$$-2[3x-2] = 9x-6$$

Distribute –2 to the quantity $[3x-2]$ to expand the left side of the equation.

$$-2[3x]-2[-2] = 9x-6$$
$$-6x+4 = 9x-6$$

Subtract $9x$ from both sides to eliminate the x-term from the right side of the equation.

$$(-6x-9x)+4 = (9x-9x)-6$$
$$-15x+4 = -6$$

The book subtracts 9x inside the equation itself this time, instead of writing -9x underneath both sides. If you prefer writing it the other way (like Problem 17.16, which adds 6x to both sides of the equation), that's fine.

Subtract 4 from both sides of the equation and isolate x.

$$-15x+(4-4) = (-6-4)$$
$$-15x = -10$$
$$\frac{-15x}{-15} = \frac{-10}{-15}$$
$$x = \frac{-10}{-15}$$

Reduce the fraction to lowest terms and remember that the quotient of two negative numbers is a positive number.

$$x = \frac{10}{15}$$
$$x = \frac{10 \div 5}{15 \div 5}$$
$$x = \frac{2}{3}$$

Note: Problems 17.18–17.19 refer to the equation $x(2x-4)+8(x+2)=2x(x-2)$.

17.18 Solve the equation for x.

Distribute x, 8, and $2x$ to their corresponding quantities and combine like terms.

$$\begin{aligned}x(2x-4)+8(x+2)&=2x(x-2)\\ [x(2x)+x(-4)]+[8(x)+8(2)]&=2x(x)+2x(-2)\\ [2x^2-4x]+[8x+16]&=2x^2-4x\\ 2x^2+(-4x+8x)+16&=2x^2-4x\\ 2x^2+4x+16&=2x^2-4x\end{aligned}$$

This is not a linear equation because it contains x^2-terms; the strategy you use to solve linear equations will not solve quadratic equations. However, if you subtract $2x^2$ from both sides of the equation, those terms are eliminated, resulting in a linear equation.

$$\begin{aligned}(2x^2-2x^2)+4x+16&=(2x^2-2x^2)-4x\\ 4x+16&=-4x\end{aligned}$$

Subtract $4x$ from both sides of the equation to group the variable terms right of the equal sign.

$$\begin{aligned}(4x-4x)+16&=-4x-4x\\ 16&=-8x\end{aligned}$$

Divide both sides of the equation by the coefficient of x to solve.

$$\begin{aligned}\frac{16}{-8}&=\frac{-8x}{-8}\\ -2&=x\end{aligned}$$

Note: Problems 17.18–17.19 refer to the equation $x(2x-4)+8(x+2)=2x(x-2)$.

17.19 Verify your solution to Problem 17.18.

Substitute $x=-2$, the solution calculated in Problem 17.18, into the equation and simplify.

$$\begin{aligned}x(2x-4)+8(x+2)&=2x(x-2)\\ -2(2[-2]-4)+8(-2+2)&=2(-2)(-2-2)\\ -2(-4-4)+8(0)&=2(-2)(-4)\\ -2(-8)+0&=-4(-4)\\ 16&=16\end{aligned}$$

Substituting $x=-2$ into the original equation results in the true statement $16=16$, which verifies the solution.

17.20 Solve the equation: $\frac{3}{4}x-7=\frac{1}{4}x-25$.

Move the x-terms to the left side of the equal sign by subtracting $\frac{1}{4}x$ from both sides of the equation.

$$\begin{aligned}\frac{3}{4}x-7&=\frac{1}{4}x-25\\ \left(\frac{3}{4}x-\frac{1}{4}x\right)-7&=\left(\frac{1}{4}x-\frac{1}{4}x\right)-25\\ \frac{2}{4}x-7&=-25\end{aligned}$$

Reduce the fraction to lowest terms.

$$\begin{aligned}\frac{2\div 2}{4\div 2}x-7&=-25\\ \frac{1}{2}x-7&=-25\end{aligned}$$

Isolate the x-term on the left side of the equal sign by adding 7 to both sides of the equation.

$$\begin{aligned}\frac{1}{2}x+(-7+7)&=-25+7\\ \frac{1}{2}x+0&=-18\\ \frac{1}{2}x&=-18\end{aligned}$$

Multiply both sides of the equation by the reciprocal of the coefficient of x.

$$\begin{aligned}\frac{2}{1}\left(\frac{1}{2}\right)x&=\frac{2}{1}\left(\frac{-18}{1}\right)\\ \frac{2}{2}x&=\frac{-36}{1}\\ x&=-36\end{aligned}$$

The coefficient of x is $\frac{1}{2}$, so multiply everything by $\frac{2}{1}$.

17.21 Solve the equation: $\frac{3}{5}(4x+30)=42$.

Distribute the fraction to the terms in parentheses.

$$\frac{3}{5}(4x+30)=42$$
$$\frac{3}{5}(4x)+\frac{3}{5}(30)=42$$

Write each of the whole numbers within parentheses as fractions and multiply.

$$\frac{3}{5}\left(\frac{4}{1}x\right)+\frac{3}{5}\left(\frac{30}{1}\right)=42$$
$$\frac{12}{5}x+\frac{90}{5}=42$$
$$\frac{12}{5}x+\frac{90\div 5}{5\div 5}=42$$
$$\frac{12}{5}x+\frac{18}{1}=42$$
$$\frac{12}{5}x+18=42$$

Isolate the x-term on the left side of the equation and eliminate its coefficient.

$$\frac{12}{5}x+(18-18)=42-18$$
$$\frac{12}{5}x=24$$
$$\frac{5}{12}\left(\frac{12}{5}\right)x=\frac{5}{12}\left(\frac{24}{1}\right)$$

Reduce the product on the right side of the equal sign before you multiply.

$$x=\frac{5}{{}_1\cancel{12}}\left(\frac{{}^2\cancel{24}}{1}\right)$$
$$x=\frac{5}{1}\left(\frac{2}{1}\right)$$
$$x=10$$

Note: Problems 17.22–17.24 refer to the equation $\frac{x}{2}=\frac{x}{3}+1$. They compare two different ways of solving equations containing fractions.

17.22 Solve the equation for x without eliminating the fractions until the final step.

In other words, don't get rid of the fractions until you're getting rid of the fractional coefficient in front of the isolated x-term.

Eliminate $\frac{x}{3}$ from the right-hand expression by subtracting it from both sides of the equation.

$$\frac{x}{2}-\frac{x}{3}=\left(\frac{x}{3}-\frac{x}{3}\right)+1$$
$$\frac{x}{2}-\frac{x}{3}=1$$

Common denominators are required to simplify the left side of the equation. Begin by expressing each term as a variable multiplied by a coefficient.

$$\frac{1}{2}x-\frac{1}{3}x=1$$

Express the difference on the left side of the equation using the least common denominator, 6.

$$\left(\frac{1}{2}\cdot\frac{3}{3}\right)x-\left(\frac{1}{3}\cdot\frac{2}{2}\right)x=1$$
$$\frac{3}{6}x-\frac{2}{6}x=1$$
$$\frac{3-2}{6}x=1$$
$$\frac{1}{6}x=1$$

To eliminate the fractional coefficient of the x-term, multiply both sides of the equation by its reciprocal, $\frac{6}{1}$.

$$\frac{6}{1}\left(\frac{1}{6}x\right)=\frac{6}{1}\cdot 1$$
$$\frac{6\cdot 1}{1\cdot 6}x=\frac{6\cdot 1}{1}$$
$$\frac{6}{6}x=\frac{6}{1}$$
$$x=6$$

Note: Problems 17.22–17.24 refer to the equation $\frac{x}{2} = \frac{x}{3} + 1$. They compare two different ways of solving equations containing fractions.

17.23 Verify your solution to Problem 17.22 using substitution.

According to Problem 17.22, $x = 6$. Substitute the solution into the original equation and simplify.

$$\frac{x}{2} = \frac{x}{3} + 1$$
$$\frac{6}{2} = \frac{6}{3} + 1$$
$$3 = 2 + 1$$
$$3 = 3$$

Because substituting $x = 6$ into the equation produces a true statement, the solution is correct.

Note: Problems 17.22–17.24 refer to the equation $\frac{x}{2} = \frac{x}{3} + 1$. They compare two different ways of solving equations containing fractions.

17.24 Solve the equation once again, but this time, begin by multiplying the entire equation by the least common denominator before you isolate x.

Rewrite the equation, expressing each x-term with an explicit coefficient.

$$\frac{1}{2}x = \frac{1}{3}x + 1$$

Multiply both sides of the equation by the least common multiple of 2 and 3, which is 6.

$$6\left(\frac{1}{2}x\right) = 6\left(\frac{1}{3}x\right) + 6(1)$$
$$\frac{6}{2}x = \frac{6}{3}x + 6$$
$$3x = 2x + 6$$

Subtract $2x$ from both sides of the equation to solve for x.

$$3x - 2x = (2x - 2x) + 6$$
$$1x = 6$$
$$x = 6$$

Because you don't have to spend any time worrying about common denominators.

Compared to the solution in Problem 17.22, fewer steps are required when fractions are eliminated early in the equation solving process.

17.25 Solve the equation: $\frac{4}{3}x-\frac{7}{9}=\frac{5}{9}x+1$.

You should eliminate the fractions from an equation before you solve it, as demonstrated in Problem 17.24. The denominators in this problem are 3 and 9, so multiply all of the terms in the equation by the least common denominator, 9.

$$\frac{4}{3}x-\frac{7}{9}=\frac{5}{9}x+1$$
$$\frac{9}{1}\left(\frac{4}{3}x\right)-\left(\frac{9}{1}\right)\left(\frac{7}{9}\right)=\frac{9}{1}\left(\frac{5}{9}x\right)+9(1)$$

Reduce the products before you multiply.

$$\frac{3\,\cancel{9}}{1}\left(\frac{4}{1\,\cancel{3}}x\right)-\left(\frac{1\,\cancel{9}}{1}\right)\left(\frac{7}{1\,\cancel{9}}\right)=\frac{1\,\cancel{9}}{1}\left(\frac{5}{1\,\cancel{9}}x\right)+9(1)$$
$$\frac{3}{1}\left(\frac{4}{1}x\right)-\left(\frac{1}{1}\right)\left(\frac{7}{1}\right)=\frac{1}{1}\left(\frac{5}{1}x\right)+9(1)$$
$$12x-7=5x+9$$

Subtract $5x$ from—and add 7 to—both sides of the equation, and solve for x.

$$(12x-5x)+(-7+7)=(5x-5x)+(9+7)$$
$$7x+0=0+16$$
$$7x=16$$
$$\frac{7x}{7}=\frac{16}{7}$$
$$x=\frac{16}{7}$$

17.26 Solve the equation: $\frac{1}{8}y+3=\frac{1}{6}y-5$.

The denominators in this equation are 8 and 6; multiply all of the terms by the least common denominator, 24.

Flip back to Problems 8.1–8.14 if you need to review least common denominators.

$$\frac{1}{8}y+3=\frac{1}{6}y-5$$
$$\left(\frac{24}{1}\right)\left(\frac{1}{8}y\right)+\left(\frac{24}{1}\right)\left(\frac{3}{1}\right)=\left(\frac{24}{1}\right)\left(\frac{1}{6}y\right)+\left(\frac{24}{1}\right)\left(\frac{-5}{1}\right)$$
$$\frac{24}{8}y+(24\cdot 3)=\frac{24}{6}y+(24)(-5)$$
$$3y+72=4y-120$$

Isolate the y-terms on the right side of the equation and the constants on the left side.

$$(3y-3y)+(72+120)=(4y-3y)+(-120+120)$$
$$0+192=1y+0$$
$$192=y$$

17.27 Solve the equation: $\frac{1}{7}\left(\frac{1}{2}x+\frac{2}{5}\right)=\frac{13}{3}x-\frac{5}{6}$.

Apply the distributive property to expand the left side of the equation.

$$\begin{aligned}\frac{1}{7}\left(\frac{1}{2}x+\frac{2}{5}\right)&=\frac{13}{3}x-\frac{5}{6}\\ \left(\frac{1}{7}\right)\left(\frac{1}{2}x\right)+\left(\frac{1}{7}\right)\left(\frac{2}{5}\right)&=\frac{13}{3}x-\frac{5}{6}\\ \frac{1}{14}x+\frac{2}{35}&=\frac{13}{3}x-\frac{5}{6}\end{aligned}$$

Identify the prime factorizations of the denominators:
$3=3$
$6=2\cdot3$
$14=2\cdot7$
$35=5\cdot7$
The LCD is the product of the unique factors: $2\cdot3\cdot5\cdot7=210$.

Multiply all of the terms by the least common denominator, 210.

$$\frac{210}{1}\left(\frac{1}{14}x\right)+\left(\frac{210}{1}\right)\left(\frac{2}{35}\right)=\left(\frac{210}{1}\right)\left(\frac{13}{3}x\right)+\left(\frac{210}{1}\right)\left(-\frac{5}{6}\right)$$

Reduce the products before you multiply.

$$\frac{15\,\cancel{210}}{1}\left(\frac{1}{1\,\cancel{14}}x\right)+\left(\frac{6\,\cancel{210}}{1}\right)\left(\frac{2}{1\,\cancel{35}}\right)=\left(\frac{70\,\cancel{210}}{1}\right)\left(\frac{13}{1\,\cancel{3}}x\right)+\left(\frac{35\,\cancel{210}}{1}\right)\left(-\frac{5}{1\,\cancel{6}}\right)$$

$$\frac{15}{1}\left(\frac{1}{1}x\right)+\left(\frac{6}{1}\right)\left(\frac{2}{1}\right)=\left(\frac{70}{1}\right)\left(\frac{13}{1}x\right)+\left(\frac{35}{1}\right)\left(-\frac{5}{1}\right)$$

$$15x+12=910x-175$$

Separate the variable terms and constants on opposite sides of the equation and solve for x.

$$\begin{aligned}(15x-15x)+(12+175)&=(910x-15x)+(-175+175)\\ 187&=895x\\ \frac{187}{895}&=\frac{895x}{895}\\ \frac{187}{895}&=x\end{aligned}$$

Proportions

Two fractions that are equal

17.28 Solve the proportion: $\frac{x}{15}=\frac{2}{5}$.

A proportion is a specific type of equation; it states that two fractions are equal, as exactly one fraction appears on each side of the equal sign. This concept is introduced in Chapter 12, where it is used to calculate percentages. In Problems 12.26–12.29, the cross-multiplication technique is introduced.

Cross-multiplication is beneficial because it changes an equation entirely constructed of fractions into an equation completely devoid of them, simply by multiplying the numerator of one fraction by the denominator of the other and setting the products equal. In this problem, to cross-multiply, you set the products $x\cdot5$ and $15\cdot2$ equal.

$$\frac{x}{15}=\frac{2}{5}$$
$$x\cdot 5=15\cdot 2$$
$$5x=30$$

Divide both sides of the equation by the coefficient of x to isolate the variable on the left side of the equal sign, and thus solve the equation.

$$\frac{5x}{5}=\frac{30}{5}$$
$$x=6$$

17.29 Solve the proportion for x: $\frac{3}{2}=-\frac{12}{x}$.

Notice that the right side of the equation is negative. Before you cross-multiply, you must assign the negative sign to either the numerator or the denominator—it does not matter which. In the solution below, the negative sign is assigned to the numerator, 12.

You can't assign the negative sign to the numerator AND the denominator, because a negative divided by a negative is a positive, and the right side of the equation is not positive.

$$\frac{3}{2}=-\frac{12}{x}$$
$$\frac{3}{2}=\frac{-12}{x}$$

Cross-multiply and solve for x.

$$3\cdot x=2(-12)$$
$$3x=-24$$
$$\frac{3x}{3}=\frac{-24}{3}$$
$$x=-8$$

17.30 Solve the equation: $\frac{x+7}{5}=3$.

This equation is not a proportion, because the expression to the right of the equal sign is not a fraction. You may express the constant 3 as a fraction and then apply cross-multiplication to solve the resulting proportion.

$$\left(3=\frac{3}{1}\right)$$

It is equally valid, however, to apply the process first introduced in Problem 17.24—multiply the entire equation by the least common denominator (5) to eliminate the fractions from the equation. The following solution below uses the this approach.

$$\frac{x+7}{5}=3$$
$$\left(\frac{5}{1}\right)\left(\frac{x+7}{5}\right)=5(3)$$
$$\left(\frac{\cancel{5}}{\cancel{5}}\right)(x+7)=15$$
$$x+7=15$$

The book doesn't show the "subtracting 7 from both sides" step—probably because the sentence above the answer explains what's happening.

Subtract 7 from both sides of the equation to solve.

$$x=8$$

17.31 Solve the proportion: $\frac{8}{x-3}=\frac{2}{3}$.

Cross-multiply to eliminate the fractions from the equation.

$$\frac{8}{x-3}=\frac{2}{3}$$
$$8\cdot 3=(x-3)\cdot 2$$
$$24=2(x-3)$$

Apply the distributive property to the right side of the equation.

$$24=2(x)+2(-3)$$
$$24=2x-6$$

Isolate the variable on the right side of the equation to solve.

$$24+6=2x+(-6+6)$$
$$30=2x$$
$$\frac{30}{2}=\frac{2x}{2}$$
$$15=x$$

17.32 Solve the proportion: $\frac{4x-3}{9}=\frac{x+5}{2}$.

Cross-multiply and apply the distributive property to the expressions on both sides of the equal sign.

$$\frac{4x-3}{9}=\frac{x+5}{2}$$
$$(4x-3)\cdot 2=9\cdot(x+5)$$
$$2(4x-3)=9(x+5)$$
$$2(4x)+2(-3)=9(x)+9(5)$$
$$8x-6=9x+45$$

If you subtract 8x from both sides, you get x on the right side. If you subtract 9x from both sides, you get −x, so you have to multiply (or divide) both sides by −1 to solve for x.

If you isolate x on the right side of the equation, it shortens your solution by one step.

$$(8x-8x)+(-6-45)=(9x-8x)+(45-45)$$
$$-51=x$$

17.33 Solve the proportion: $\frac{x-2}{5x-1}=\frac{2}{7}$.

Cross-multiply and apply the distributive property on both sides of the equation.

$$\begin{aligned}\frac{x-2}{5x-1}&=\frac{2}{7}\\(x-2)\cdot 7&=(5x-1)\cdot 2\\7(x-2)&=2(5x-1)\\7(x)+7(-2)&=2(5x)+2(-1)\\7x-14&=10x-2\end{aligned}$$

Isolate x to solve the equation.

$$\begin{aligned}(7x-7x)+(-14+2)&=(10x-7x)+(-2+2)\\-12&=3x\\\frac{-12}{3}&=\frac{3x}{3}\\-4&=x\end{aligned}$$

17.34 Solve the proportion: $-\frac{3x}{8}=\frac{x+4}{2}$.

As explained in Problem 17.29, when one (or both) of the fractions in a proportion is negative, the negative sign must be assigned to either the numerator or the denominator of that fraction before you cross-multiply. In the solution below, the negative sign on the left side of the equal sign is assigned to the numerator, changing it from $3x$ to $-3x$.

$$\begin{aligned}-\frac{3x}{8}&=\frac{x+4}{2}\\\frac{-3x}{8}&=\frac{x+4}{2}\end{aligned}$$

Cross-multiply and solve for x.

$$\begin{aligned}-3x(2)&=8(x+4)\\-3x(2)&=8(x)+8(4)\\-6x&=8x+32\\(-6x-8x)&=(8x-8x)+32\\-14x&=32\\\frac{-14x}{-14}&=\frac{32}{-14}\\x&=-\frac{32}{14}\end{aligned}$$

Reduce the fraction to lowest terms.

$$\begin{aligned}x&=-\frac{32\div 2}{14\div 2}\\x&=-\frac{16}{7}\end{aligned}$$

Solving for a Specific Variable

Same process, just more variables in the equation

Note: Problems 17.35–17.36 refer to the following equation: $rs = 10t$.

17.35 Solve the equation for t.

There are lots of reasons it's not a linear equation, but one big reason is that it contains more than one variable.

Although $rs = 10t$ is not a linear equation, you use the same techniques described in the preceding problems to isolate a particular variable on one side of the equal sign. To solve an equation for a specific variable, complete the following steps:

1. Simplify both sides of the equation (if necessary).
2. Use the addition or subtraction property of equality to move all of the terms containing the variable for which you are solving to one side of the equation.
3. Combine like terms. The end result should be a term containing the variable for which you are solving on one side of the equation. The other side of the equation has no specific form—it may vary.
4. Eliminate the numbers and variables multiplied by the variable for which you are solving. Usually, this is accomplished by dividing both sides of the equations by those values. If, however, a fraction is involved, you may need to multiply both sides of the equation by the reciprocal. (See Problem 17.6.)

This problem instructs you to solve $rs = 10t$ for t. Notice that the term containing t is already isolated on the right side of the equal sign. In order to solve for t, all you must do is eliminate its coefficient, 10. Divide both sides of the equation by 10.

$$\begin{aligned} rs &= 10t \\ \frac{rs}{10} &= \frac{10t}{10} \\ \frac{rs}{10} &= 1t \\ \frac{rs}{10} &= t \end{aligned}$$

The answer $\frac{1}{10}rs = t$ is also correct, because the product rs may be expressed as a fraction with a denominator of 1, as demonstrated below.

$$\frac{1}{10}rs = \frac{1}{10} \cdot \frac{rs}{1} = \frac{rs}{10}$$

Note: Problems 17.35–17.36 refer to the following equation: rs = 10t.

17.36 Solve the equation for r.

The term rs, which contains the variable for which you are solving (r), is already isolated on the left side of the equation. In Problem 17.35, you eliminate a *constant* using the division property of equality. Use the same strategy to eliminate a *variable* from one side of the equation. In other words, divide both sides of the equation by s to solve for r.

$$rs = 10t$$
$$\frac{rs}{s} = \frac{10t}{s}$$
$$r = \frac{10t}{s}$$

Therefore, the correct solution is $r = \frac{10t}{s}$, $s \neq 0$. Note that $r = 10\frac{t}{s}$, $s \neq 0$ and $r = 10t\frac{1}{s}$, $s \neq 0$ are also acceptable solutions.

Technically, you can't divide by s unless you assume that s cannot equal zero. You're not allowed to divide by zero. It's a technicality, but that's why the book includes "$s \neq 0$" with the final answer.

Note: Problems 17.37–17.38 refer to the following equation: 3x – y = 6z.

17.37 Solve the equation for y.

Isolate the term containing y on the left side of the equation by subtracting $3x$ from both sides.

$$\begin{aligned} 3x - y &= 6z \\ (3x - 3x) - y &= 6z - 3x \\ -y &= 6z - 3x \end{aligned}$$

The equation is now solved for $-y$. To solve it for y, multiply both sides of the equation by -1.

$$\begin{aligned} -1(-y) &= -1(6z - 3x) \\ -1(-y) &= (-1)(6z) + (-1)(-3x) \\ y &= -6z + 3x \end{aligned}$$

Although the above solution is acceptable, mathematics convention dictates that you list the terms in alphabetical order, according to their variables. Thus, the preferred solution reverses the terms: $y = 3x - 6z$.

Note: Problems 17.37–17.38 refer to the following equation: $3x - y = 6z$.

17.38 Solve the equation for x.

Add y to both sides of the equation to isolate the x-term to the left of the equal sign.

$$3x - y = 6z$$
$$3x + (-y + y) = 6z + y$$
$$3x = y + 6z$$

Divide both sides of the equation by 3, so that only x remains on the left side of the equation.

$$\frac{3x}{3} = \frac{y + 6z}{3}$$
$$x = \frac{y + 6z}{3}$$

You can write this as the sum of two separate fractions if you want to:
$x = \frac{y}{3} + \frac{6z}{3}$
However, you should divide 6 by 3 and get 2 to simplify the fraction on the right:
$x = \frac{y}{3} + 2z$

Alternatively, you may multiply both sides of the equation $3x = y + 6z$ by $\frac{1}{3}$ to eliminate the x-coefficient. If you choose this method, it produces the equally valid solution $x = \frac{1}{3}(y + 6z)$.

17.39 Solve the equation below for c.

$$d = b^2 - 4ac$$

Isolate the term containing c by subtracting b^2 from both sides of the equation.

$$d - b^2 = (b^2 - b^2) - 4ac$$
$$d - b^2 = -4ac$$

You are asked to solve for c, and it is multiplied by -4 and a within the term $-4ac$. Divide by those two values to solve for c.

$$\frac{d - b^2}{-4a} = \frac{-4ac}{-4a}$$
$$-\frac{d - b^2}{4a} = c$$
$$-\frac{-b^2 + d}{4a} = c,\ a \neq 0$$

If you choose to assign the negative sign to the numerator of the fraction on the left side of the fraction, it produces the equivalent solution below.

$$\frac{-(-b^2+d)}{4a}=c$$
$$\frac{-(-b^2)-(d)}{4a}=c$$
$$\frac{b^2-d}{4a}=c,\ a\neq 0$$

17.40 According to Problem 12.30, you can calculate a percentage c given a portion p of a whole w by applying the formula below. Solve the proportion for c so that you can quickly calculate a percentage c—assuming that you know the values of p and w—without needing to cross-multiply each time you apply the formula.

$$\frac{c}{100}=\frac{p}{w}$$

Cross-multiply and divide by any factors that are multiplied by c in order to solve for c.

$$\frac{c}{100}=\frac{p}{w}$$
$$cw=100p$$
$$\frac{cw}{w}=\frac{100p}{w}$$
$$c=\frac{100p}{w},\ w\neq 0$$

The solution $c=100\frac{p}{w},\ w\neq 0$ is also a valid solution.

17.41 Solve the proportion below for b.

$$\frac{x}{y}=\frac{4a+c}{a+b}$$

Cross-multiply to eliminate the fractions.

$$\frac{x}{y}=\frac{4a+c}{a+b}$$
$$x(a+b)=y(4a+c)$$
$$x(a)+x(b)=y(4a)+y(c)$$
$$ax+bx=4ay+cy$$

Subtract ax from both sides of the equation in order to isolate the term containing b left of the equal sign.

$$(ax-ax)+bx=4ay+cy-ax$$
$$bx=4ay+cy-ax$$

Divide both sides of the equation by x to solve for b.

$$\frac{bx}{x}=\frac{4ay+cy-ax}{x}$$
$$b=\frac{4ay+cy-ax}{x},\ x\neq 0$$

Chapter 18

CARTESIAN COORDINATE SYSTEM

A quick introduction to graphing

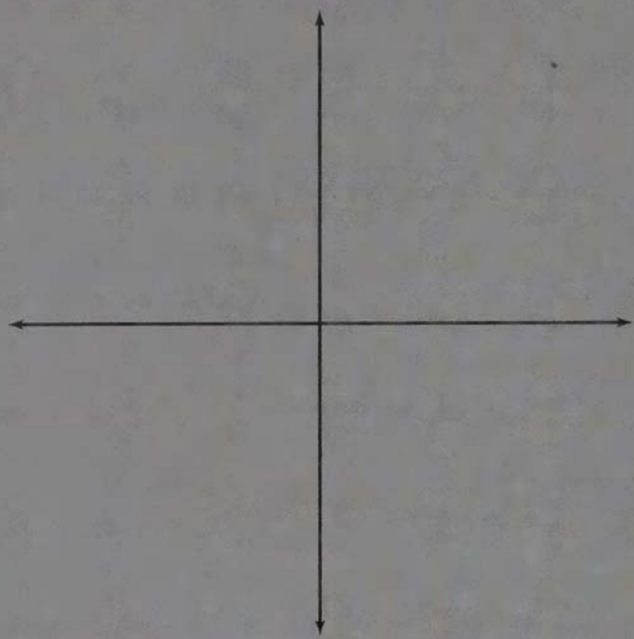
The Cartesian coordinate system allows you to visualize equations graphically in a two-dimensional plane using coordinates, pairs of x- and y-values that represent points in the plane. Collections of these coordinates are called graphs, so the process of plotting collections of related points is described as graphing. In this chapter, you will explore the components of the coordinate system, plot points, and investigate the graphs of linear equations.

This chapter is very unique. It provides a basic introduction to the world of graphs by first examining the pieces of a coordinate system that act as the skeleton to hold the muscle of points and graphs. After you understand how the coordinate system works, you'll start plotting points, and before long, you'll be graphing lines. You'll even put your equation solving skills from Chapter 17 to work.

Elements of a Graph

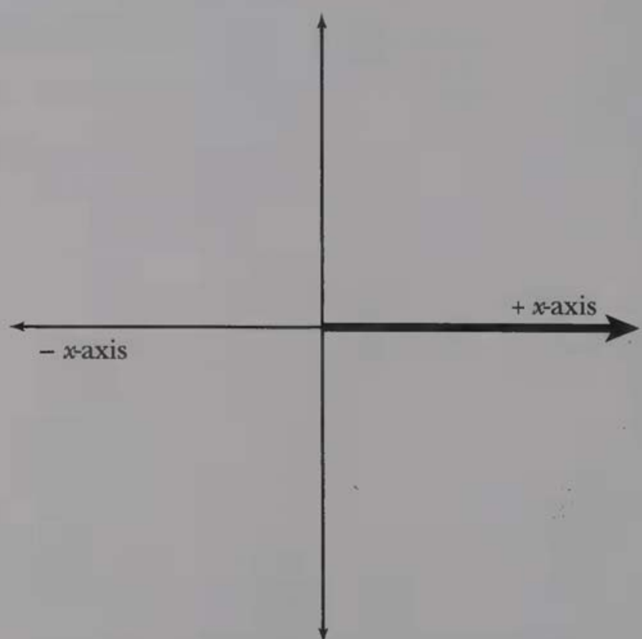
Axes, quadrants, and the origin

18.1 Identify the x-axis in the illustration below and darken the portion of the x-axis that represents positive x-values.

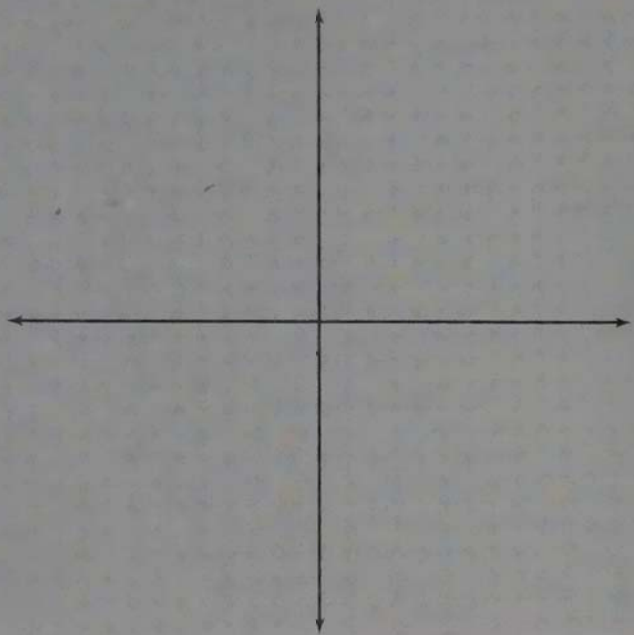


A Cartesian coordinate system consists of two axes, each representing one dimension of the two-dimensional system. Each dimension is described using a different variable, so each axis is named after that variable. A typical coordinate system has a horizontal axis, which is referred to as the x-axis.

Positive values of x are plotted on the right half of the x-axis and negative values of x are plotted on the left half. This problem asks you to identify the positive portion of the x-axis, so darken the right half of the x-axis, as demonstrated below.

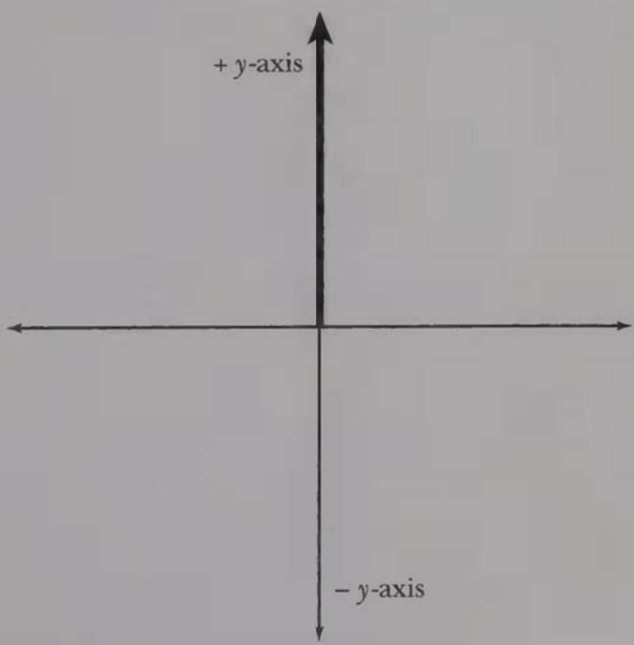


18.2 Identify the y-axis in the illustration below and shade the portion of the y-axis that represents positive y-values.

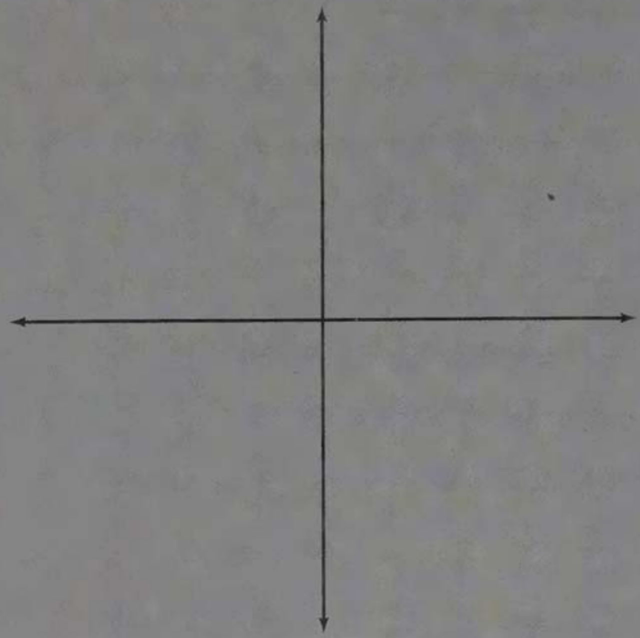


Problem 18.1 explains that the horizontal axis of a typical Cartesian coordinate system represents the x-axis. Hence, the vertical axis represents the remaining dimension of the two-dimensional system, and it is called the y-axis. The y-axis is perpendicular to the x-axis, and at that point of intersection, it—like the x-axis—is divided into positive and negative portions.

The half of the y-axis appearing above the x-axis represents positive values of y, so it is darkened in the diagram below, as requested by this problem. The half of the y-axis lying below the x-axis represents negative values of y.

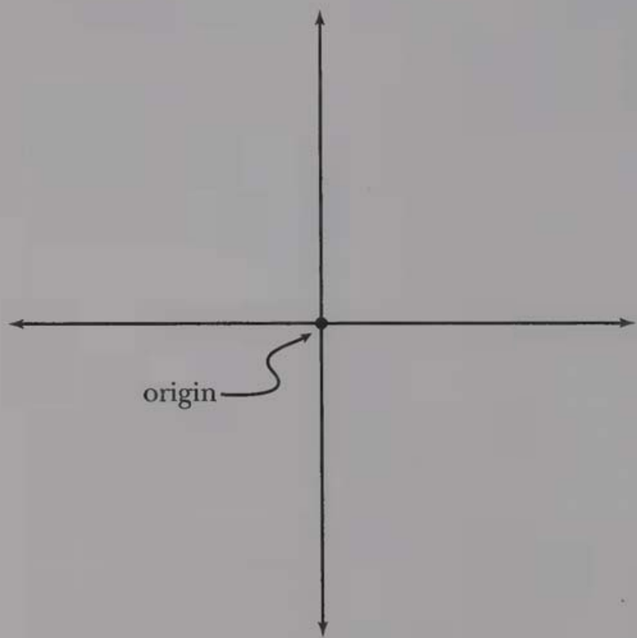


18.3 Identify the origin in the Cartesian coordinate system below.



Problems 18.1–18.2 identify the x- and y-axes of a Cartesian coordinate system. Those axes intersect at exactly one point, where the lines are perpendicular to one another. The point at which the axes intersect is called the *origin*, as illustrated in the diagram below.

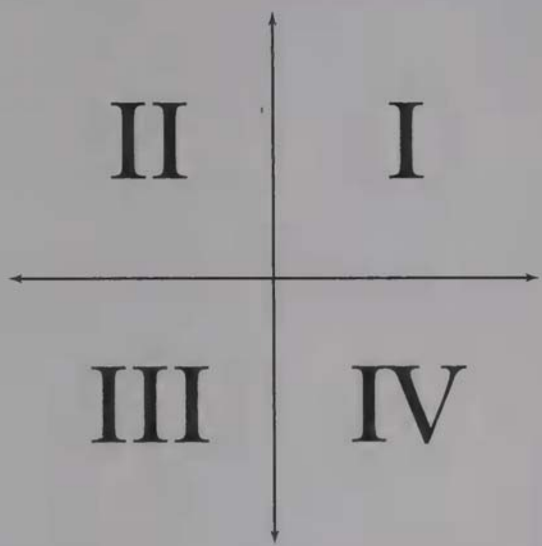
Perpendicular lines form 90°, or right, angles. Look at Problems 19.2–19.3 for more information.



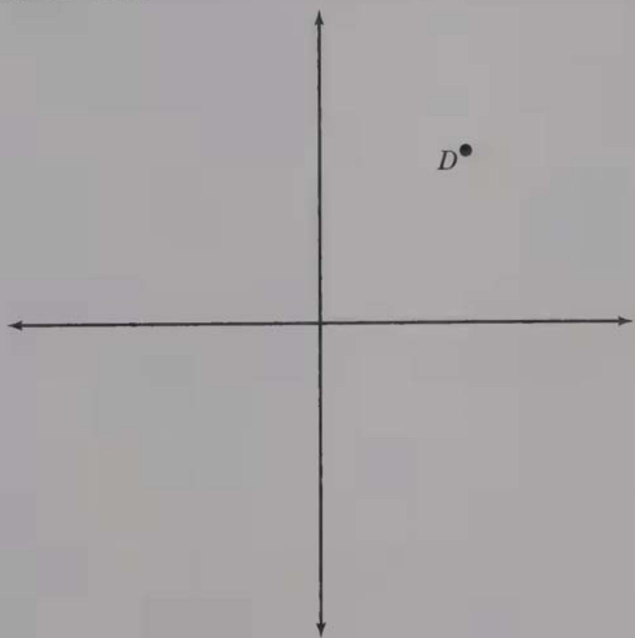
18.4 Plot a point D that lies in Quadrant I of a Cartesian coordinate system.

The x- and y-axes of the Cartesian coordinate system divide the two-dimensional plane into four distinct regions called *quadrants*. The quadrants are numbered for the purpose of reference in a very specific way. Most students assume that the quadrants are numbered beginning with the upper-left region. However, this is not true. The first quadrant is the upper-right region of the plane, and you proceed from there in a counter-clockwise direction. The diagram below identifies the quadrants by number.

And, for some reason, they are usually labeled with Roman numerals.

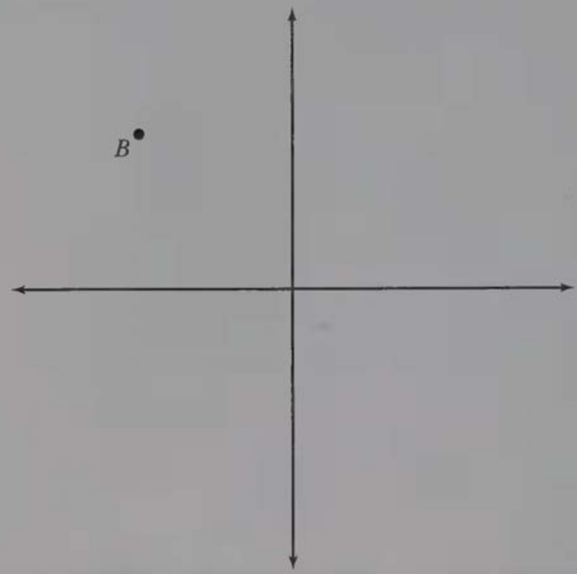


To complete this problem, plot a point D anywhere in the upper-right region of the coordinate plane—the portion of the plane bounded by (but not including) the positive x- and y-axes, the darkened portions of the axes that you identified in Problems 18.1–18.2.



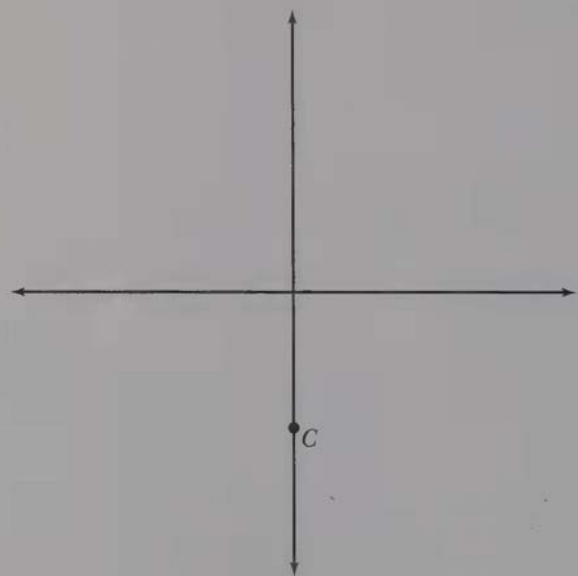
18.5 Plot a point B that lies in Quadrant II of a Cartesian coordinate system.

Consider the visual aid presented in Problem 18.4, which labels each of the four quadrants in the coordinate plane by number. The second quadrant—also called Quadrant II—is the upper-left region of the plane. It is bounded on the right by the positive y-axis, and it is bounded below by the negative x-axis. To complete this problem, plot a point labeled B that lies in Quadrant II, as demonstrated below.



18.6 Plot a point C that does not lie in Quadrants I, II, III, or IV, and is not the origin.

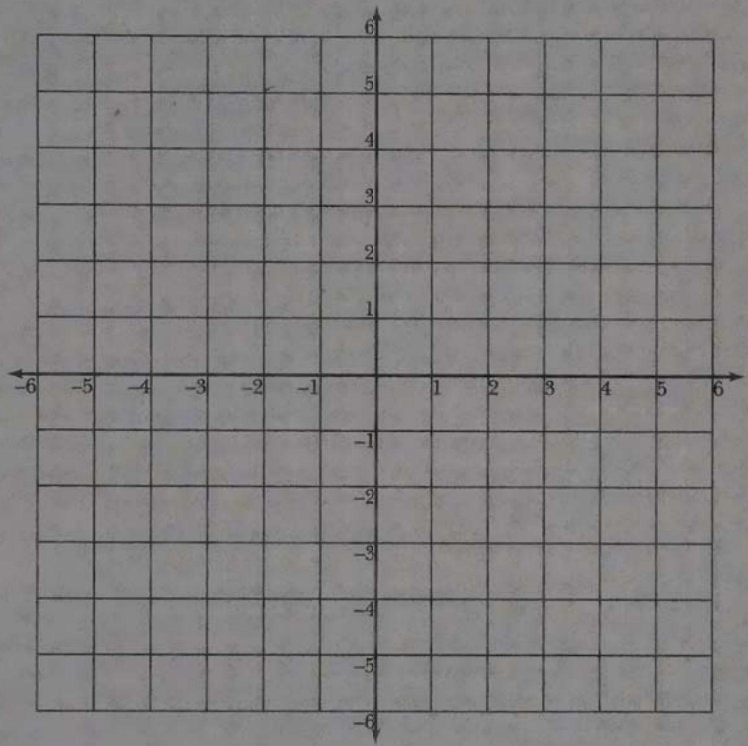
Because the x- and y-axes bound the quadrants, but do not actually belong to the quadrants, *any* point that lies on either axis (other than the origin) satisfies the conditions of this problem. One such point appears in the sample solution below.



Plotting Points

x = horizontal, y = vertical

18.7 Plot the point (4,1) on the Cartesian coordinate system below and explain your work.

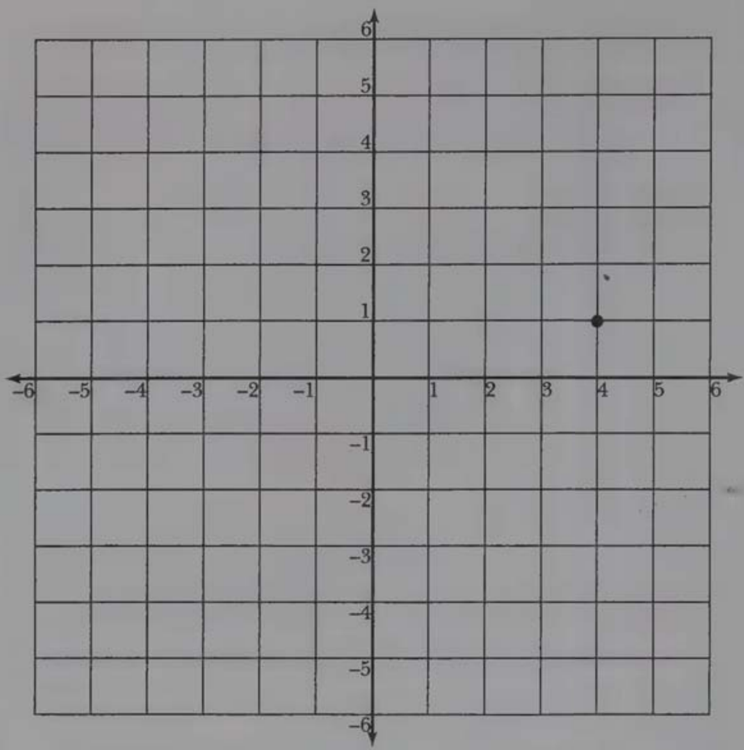


A point on the Cartesian coordinate system is identified by two coordinates within parentheses. They take the form (x,y); the x-coordinate is listed first, followed by the y-coordinate. The point (4,1) has positive x-coordinate 4. As Problem 18.1 explains, points with positive values of x should be graphed on the right side of the x-axis. Beginning at the origin, move four grid lines to the right. The point will lie along the vertical line labeled 4.

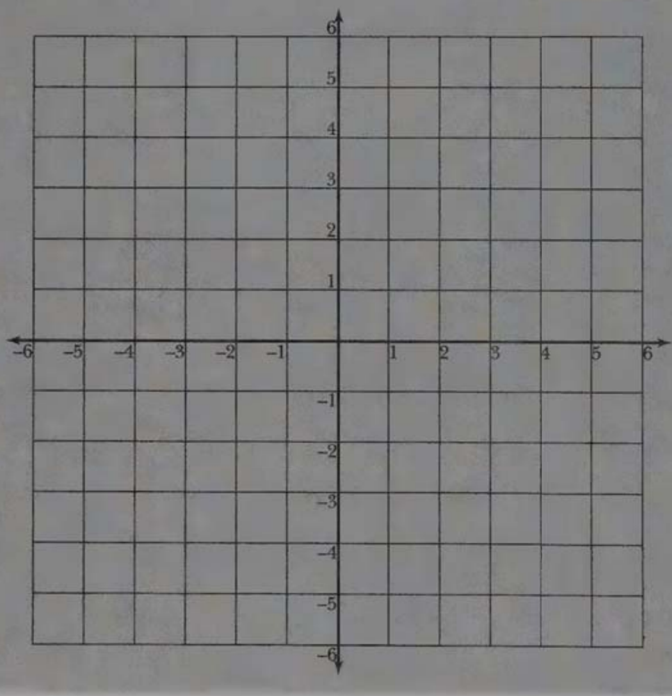
The point has positive y-coordinate 1. As Problem 18.2 explains, points with positive values of y should be graphed along the upper portion of the y-axis, above the x-axis. From the origin, move one horizontal line upwards. The point (4,1) will lie along this line.

Therefore, (4,1) is the point at which the vertical line located four units right of the origin intersects the horizontal line located one unit above the origin, as illustrated in the following diagram.

The point is above the x-axis and right of the y-axis, so it appears in Quadrant I.

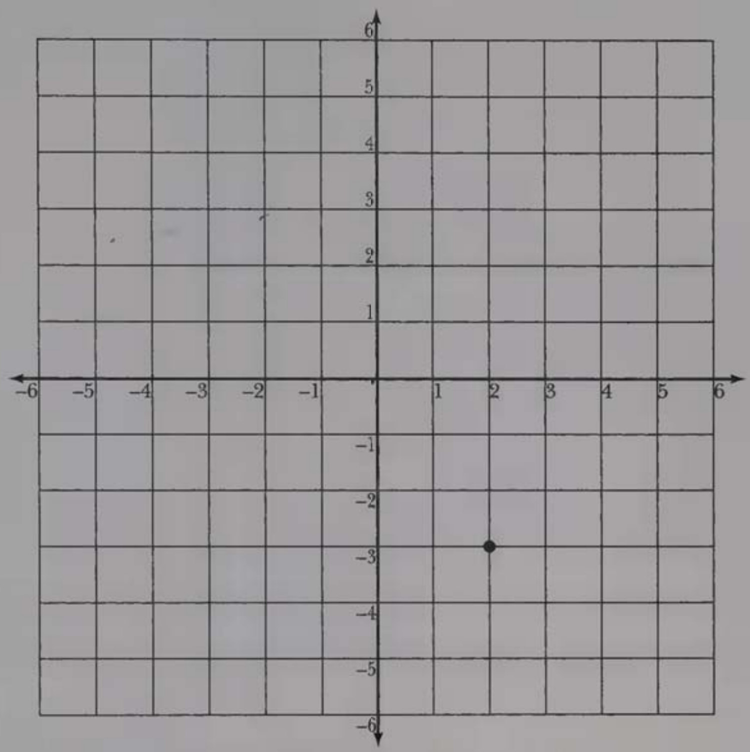


18.8 Plot the point (2,–3) on the Cartesian coordinate system below and explain your work.

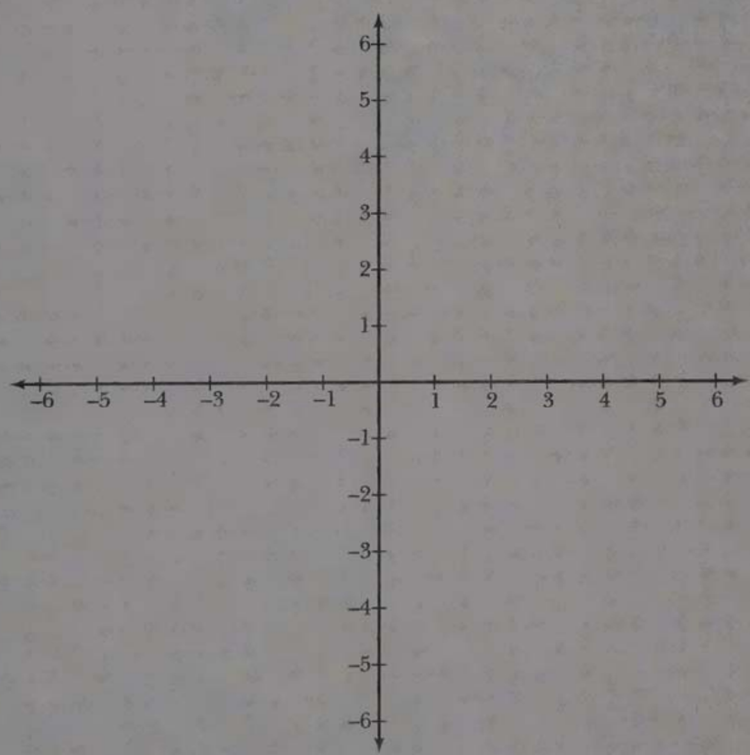


The x-coordinate of a point dictates whether the point will be located right or left of the origin. This point has positive x-coordinate 2; so the point lies along the vertical line two units to the right of the origin. The y-coordinate of a point determines whether the point will be located above or below the origin. This point has negative y-coordinate –3, so the point lies on the horizontal line three units below the origin.

Plot point (2,–3) at the intersection of vertical line $x = 2$ and horizontal line $y = -3$.



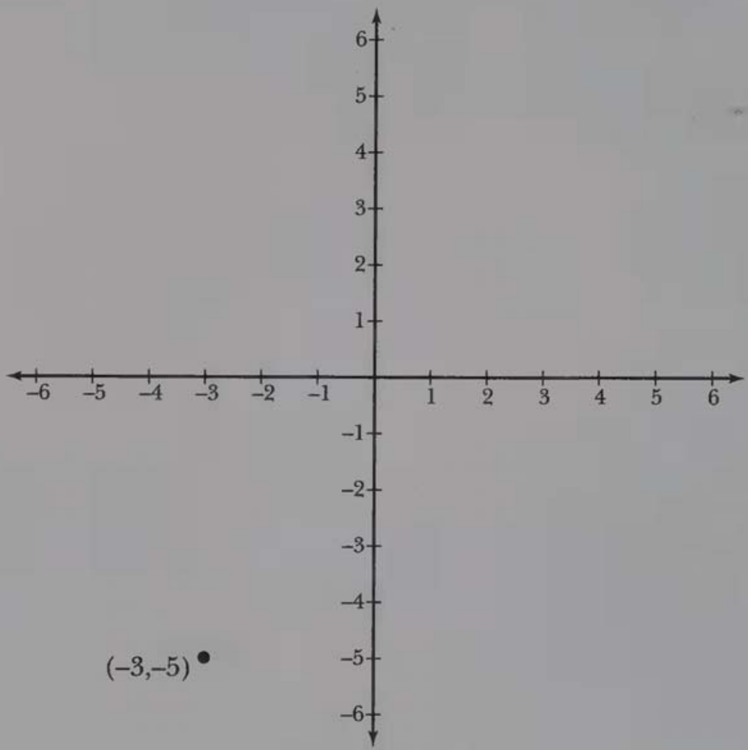
18.9 Plot the point (–3,–5) on the Cartesian coordinate system below.



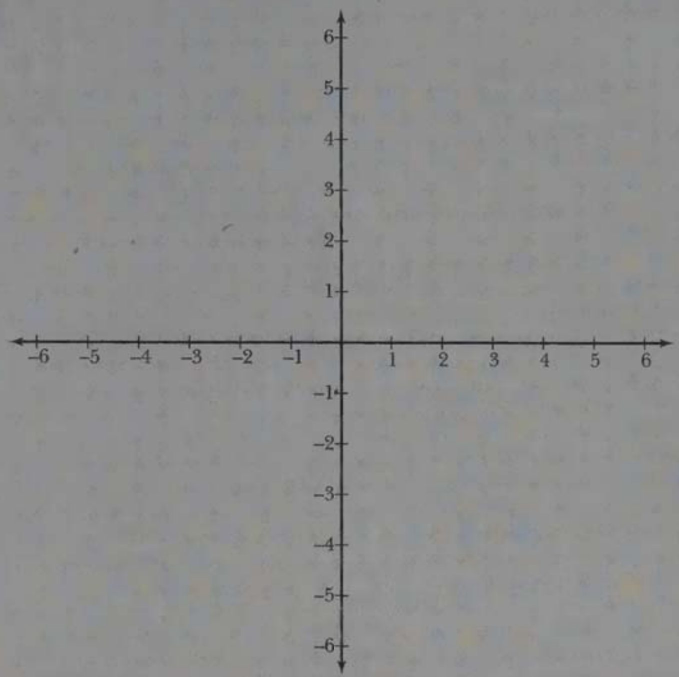
The coordinate system in this problem looks slightly different than the systems in Problems 18.7–18.8; the grid lines are omitted. While you may find graph paper to be a useful tool as you begin graphing, once you are comfortable with the process, you can construct your own axes with regularly spaced marks along the x- and y-axes to represent units (instead of full grid lines). The process, however, remains the same.

Positive x: move right
Negative x: move left
Positive y: move up
Negative y: move down

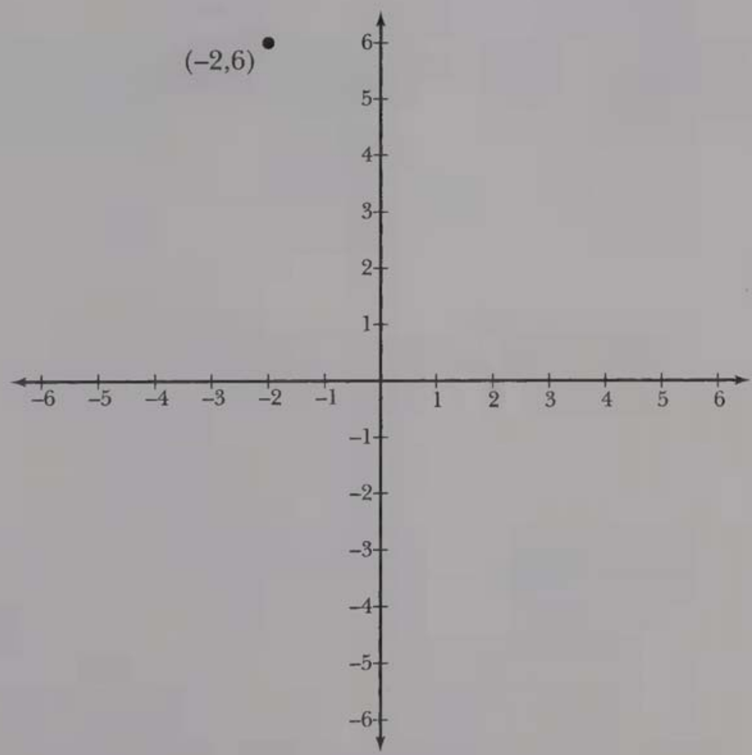
The x-coordinate is –3, so the point is located on the vertical line three units left of the origin. The y-coordinate is –5, so the point is also located on the horizontal line five units below the origin. Plot the point at the intersection of the specified vertical and horizontal lines, as illustrated below.



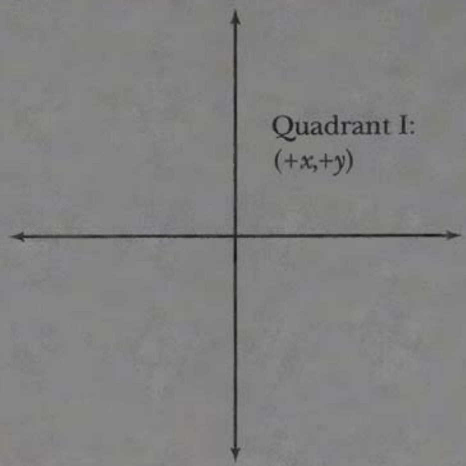
18.10 Plot the point (–2,6) on the Cartesian coördinate system below.



To plot a point, begin at the origin and travel the horizontal and vertical distances specified by the x- and y-coordinates. The point in this problem is located two units left of, and six units above, the origin.



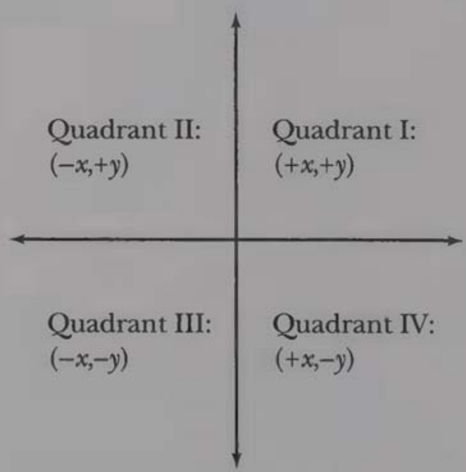
18.11 In the diagram below, Quadrant I is labeled $(+x,+y)$ because the x- and y-coordinates of all points in Quadrant I are positive. Complete the diagram by identifying the signs of the coordinates in the remaining three quadrants.



Each quadrant is bounded by portions of the x- and y-axes, which define the signs of the values within the regions as follows:

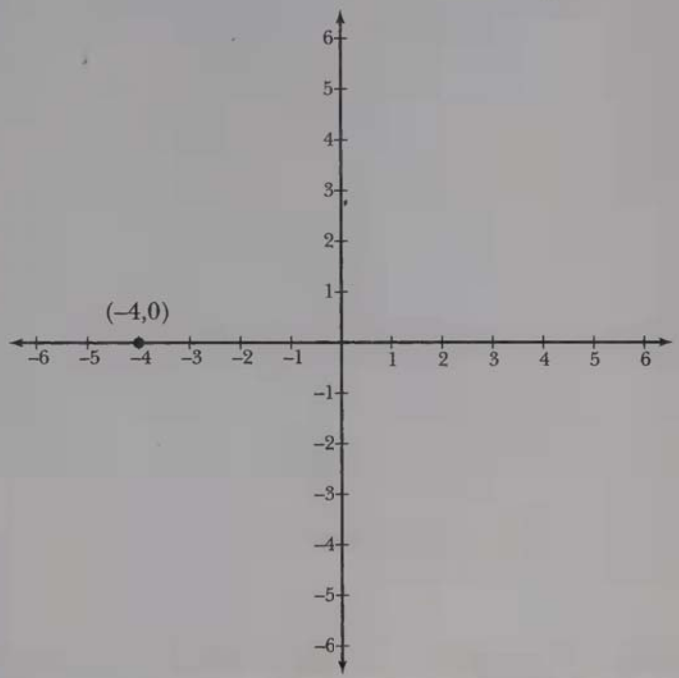
- Quadrants I and IV are located right of the y-axis, along the positive portion of the x-axis. Therefore, points within these regions have positive x-coordinates.
- Quadrants II and III are located left of the y-axis, along the negative portion of the x-axis, so they contain points with negative x-coordinates.
- Quadrants I and II are above the x-axis, along the positive portion of the y-axis, so the points within those regions have positive y-coordinates.
- Quadrants III and IV are below the x-axis, along the negative portion of the y-axis, so points within these quadrants have negative y-coordinates.

The first two bullets in the preceding list define the signs of the x-values for all four quadrants; the final two bullets define the signs of the y-values. Use the information in the list to complete the diagram.



18.12 Plot the point (–4,0) on a Cartesian coordinate system.

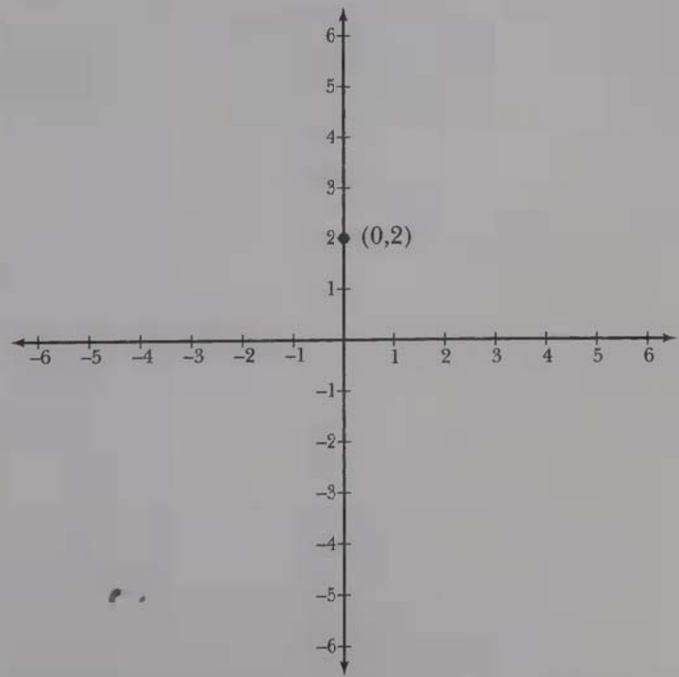
This problem does not provide a system upon which to plot the point. Create your own coordinate system using graph paper or by drawing evenly spaced marks along the axes. The solution below opts for the latter coordinate system. The point (–4,0) is four units left of the origin and zero units above or below it. The point lies on the *x*-axis, four units left of the origin.



18.13 Plot the point (0,2) on a coordinate plane.

The *x*-coordinate of this point is zero, so you do not travel any horizontal distance. According to the *y*-coordinate, the point should be located exactly two units above the origin along the *y*-axis.

"Coordinate plane" and "coordinate system" mean the same thing. "Plane" just means "flat surface that holds the graph"—in other words, the sheet of paper you draw the axes on.

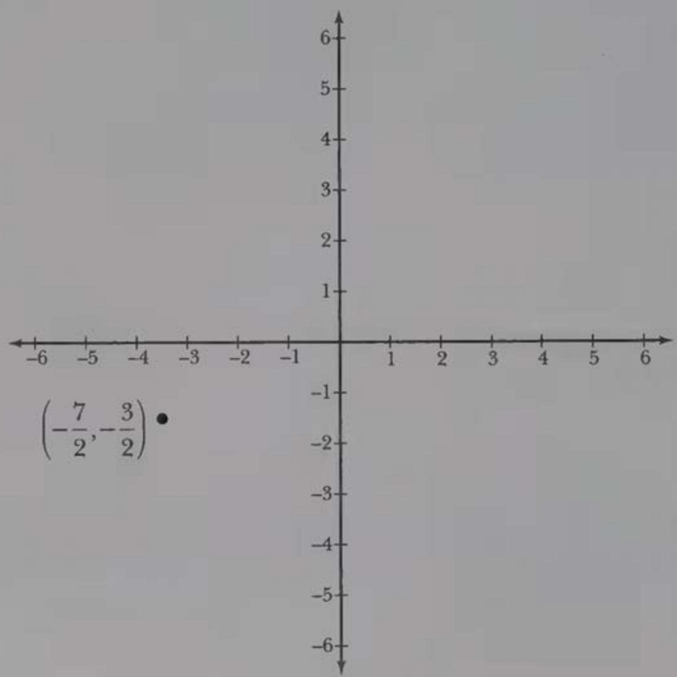


18.14 Plot the point $\left(-\frac{7}{2},-\frac{3}{2}\right)$ on a coordinate plane.

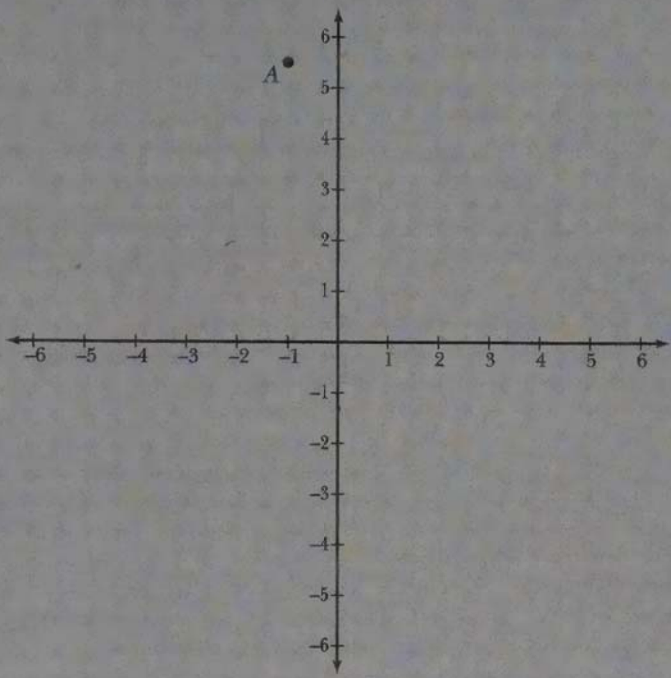
It may help you to convert the improper fractions to mixed numbers, using the technique first described in Problem 7.11.

$$-\frac{7}{2}=-3\frac{1}{2} \qquad -\frac{3}{2}=-1\frac{1}{2}$$

Plot the point $3\frac{1}{2}$ units left of, and $1\frac{1}{2}$ units below, the origin. The point should lie halfway between –3 and –4 horizontally and halfway between –1 and –2 vertically.



18.15 Identify the coordinates of point A in the graph below.



The point lies on the vertical line one unit left of the y-axis and $5\frac{1}{2}$ units above the x-axis. It is also acceptable to write the y-coordinate as the equivalent improper fraction $\frac{11}{2}$. Thus, $A = \left(-1, 5\frac{1}{2}\right)$ or $A = \left(-1, \frac{11}{2}\right)$.

Equations of Horizontal and Vertical Lines

"x =" means vertical, "y =" means horizontal

18.16 Identify the unique characteristic that all points on the x-axis share, and use it to identify the equation that represents the x-axis.

In Problem 18.12, you graph the point $(-4,0)$ on the x-axis, four units left of the origin. Note that the y-coordinate of the point is 0. Because you move zero units in the horizontal direction—you neither move up nor down from the x-axis once you have accounted for the x-coordinate -4 by moving four units left of the origin—the point lies on the x-axis.

In fact, all points with a y-coordinate of 0 lie on the x-axis. Therefore, the equation of the x-axis is $y = 0$.

So the equation of the x-axis doesn't have an x in it, only a y.

18.17 Based upon your answer to Problem 18.16, identify the equation that represents the y-axis and explain your answer.

According to Problem 18.16, the equation of the x-axis is $y = 0$. It is reasonable to hypothesize, therefore, that reversing the variables also produces a true statement: The equation of the y-axis is $x = 0$. Not only is it reasonable, it is also mathematically correct. Each of the points on the y-axis must have an x-coordinate of 0, because signed (positive and negative) x-coordinates would move the points in one of the quadrants, as explained in Problem 18.11.

Just as all points on the x-axis must have a y-coordinate of 0, all points on the y-axis must have an x-coordinate of 0. Hence, a point that lies on the y-axis satisfies the equation $x = 0$.

18.18 Based upon your answers to Problems 18.16–18.17, draw a general conclusion about the equations of horizontal and vertical lines.

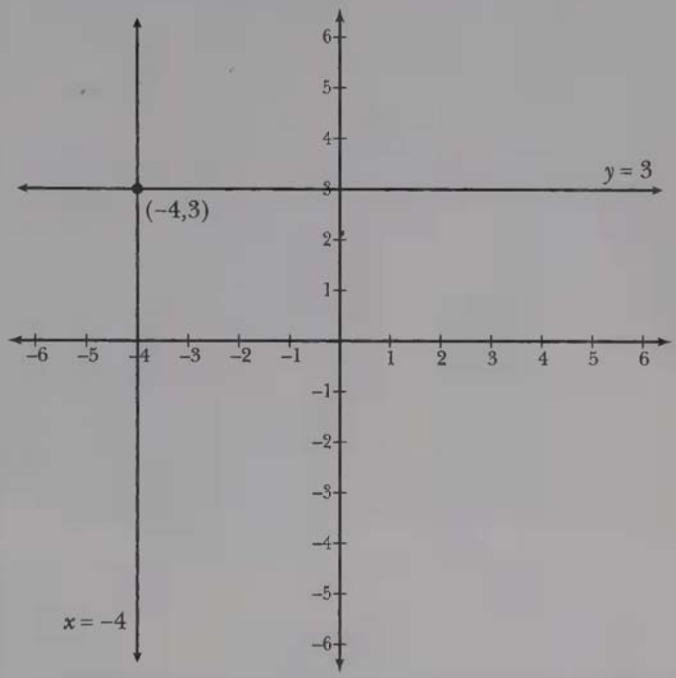
According to Problems 18.16, the horizontal x-axis has equation $y = 0$. By extension, all horizontal lines have equation $y = d$, where d is its vertical distance from the x-axis. For example, a horizontal line that is one unit above—and parallel to—the x-axis has equation $y = 1$. The horizontal line one unit below the x-axis has equation $y = -1$.

Problem 18.17 explains that the vertical y-axis has equation $x = 0$. The equations of all vertical lines take the form $x = d$, where d is the horizontal distance between the vertical line and the y-axis. For example, the vertical line five units left of—and parallel to—the y-axis has equation $x = -5$. The vertical line five units right of the y-axis has equation $x = 5$.

Pick one point on the y-axis, say $(0, -6)$ for example. It has coordinates $x = 0$ and $y = -6$. The equation of the y-axis is $x = 0$. If you substitute the x-value of this point into the equation, you get $0 = 0$, which is true. That means $(0, -6)$ lies on the line $y = 0$.

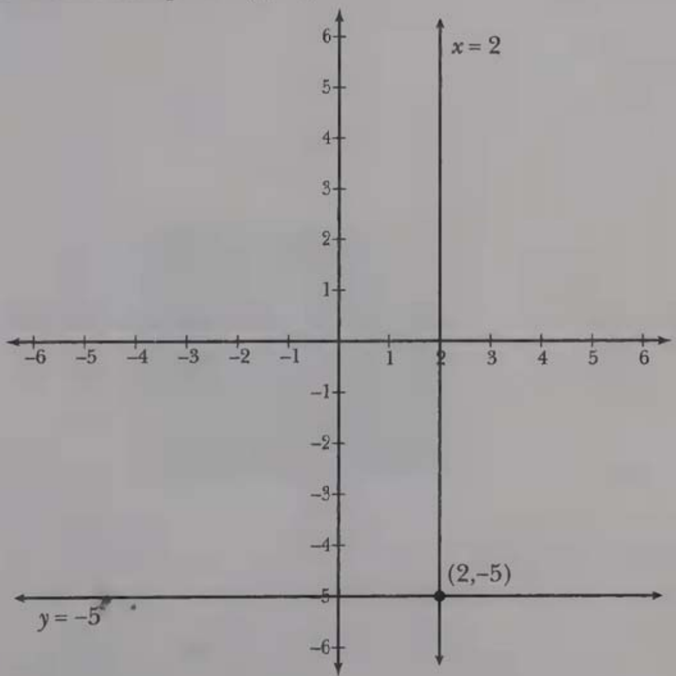
18.19 Identify the equations of the horizontal and vertical lines that contain the point $P = (-4,3)$. Graph the lines and plot the point on a Cartesian coordinate plane.

The x-coordinate of P is -4, which lies on the vertical line $x = -4$. The y-coordinate of P is 3, which lies on horizontal line $y = 3$.

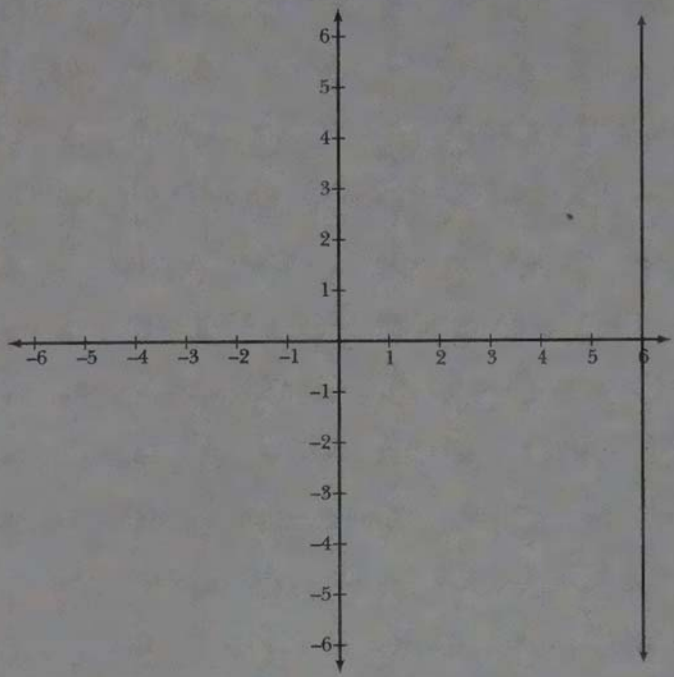


18.20 Graph the lines $x = 2$ and $y = -5$. Identify the point at which the lines intersect.

The graph of $x = 2$ is a vertical line two units right of the y-axis; the graph of $y = -5$ is a horizontal line five units below the x-axis. The graphs of the linear equations intersect at point $(2,-5)$.



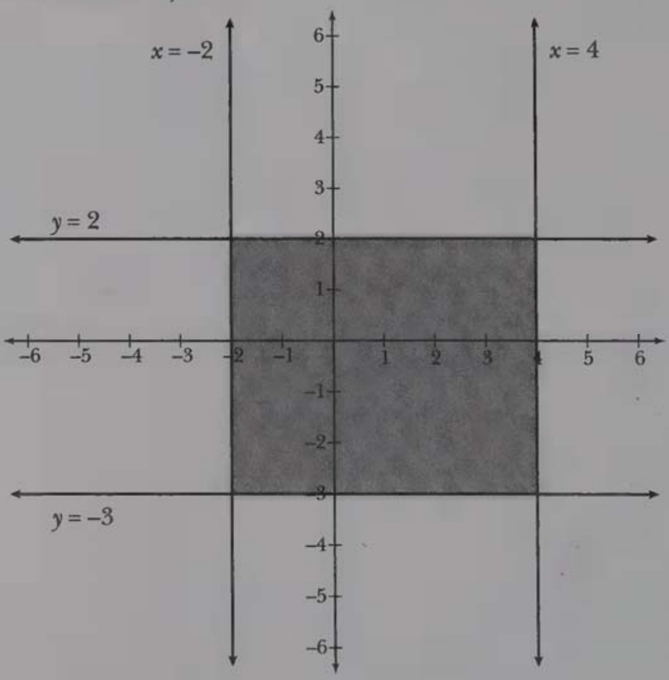
18.21 Identify the equation of the line graphed below.



All vertical lines have equation $x = d$, where d is the horizontal distance between the line and the y-axis. This vertical line is six units right of the y-axis, so the equation is $x = 6$.

18.22 Shade the rectangular region bounded by the following four lines: $y = -3$, $x = -2$, $y = 2$, and $x = 4$.

The rectangular shaded region is bounded above by horizontal line $y = 2$ and below by $y = -3$. The left side of the rectangle lies on vertical line $x = -2$ and the right side is bounded by $x = 4$.



Graphing Lines Using Substitution

Plug in a few x's to make points

Note: In Problems 18.23–18.24, you create a table of values to graph linear equation $y = x + 1$.

18.23 Substitute x-values –2, 0, and 3 into the equation to calculate corresponding values of y.

A table of values is a chart in which the left column contains the x-values that you substitute into an equation. The substitution is completed in the right column. Each value of x that you substitute into a linear equation corresponds with exactly one value of y. Substitute $x = -2$, $x = 0$, and $x = 3$ into the equation, using separate rows in the table for each value of x, and calculate y in each row.

x	$y = x + 1$
-2	$y = -2 + 1$ $= -1$
0	$y = 0 + 1$ $= 1$
3	$y = 3 + 1$ $= 4$

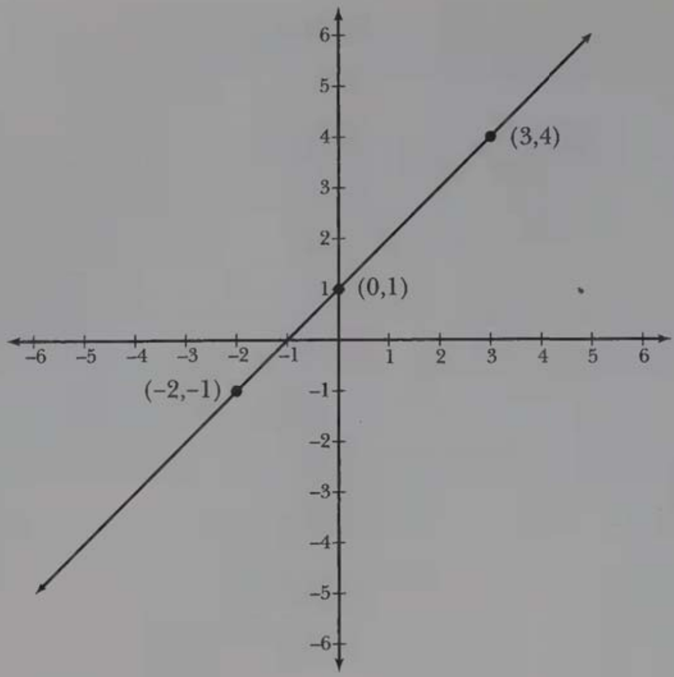
Note: In Problems 18.23–18.24, you create a table of values to graph linear equation $y = x + 1$.

18.24 Construct points using the pairs of x- and y-values you generated in Problem 18.23 and plot them on a coordinate plane to graph the linear equation.

According to Problem 18.23, substituting $x = -2$ into the equation $y = x + 1$ produces the corresponding y-value –1. Therefore, the point $(x,y) = (-2,-1)$ belongs to the graph of $y = x + 1$. By the same logic, the points (0,1) and (3,4) also belong to the graph.

Technically, you only need two points to draw the graph of a line. It's a good idea to use three points, however, to guard against arithmetic mistakes. One, straight line should pass through all three points.

Plot all three points on a coordinate plane and draw a line through them to graph the equation. The equation $y = x + 1$ is a linear equation because all of the points on its graph lie on the same line in the coordinate plane, as illustrated in the following graph.



Note: In Problems 18.25–18.26, you create a table of values to graph linear equation $y=-\frac{1}{2}x+3$.

18.25 Substitute x-values −4, 0, and 2 into the equation to calculate corresponding values of y.

Substitute $x=-4$, $x=0$, and $x=2$ into the equation and calculate the corresponding value of y for each. Note that even values of x are chosen, because the equation requires you to calculate half of x. Choosing odd numbers for the table of values will result in fractional coordinates and add unnecessary difficulty to the task of graphing in Problem 18.26.

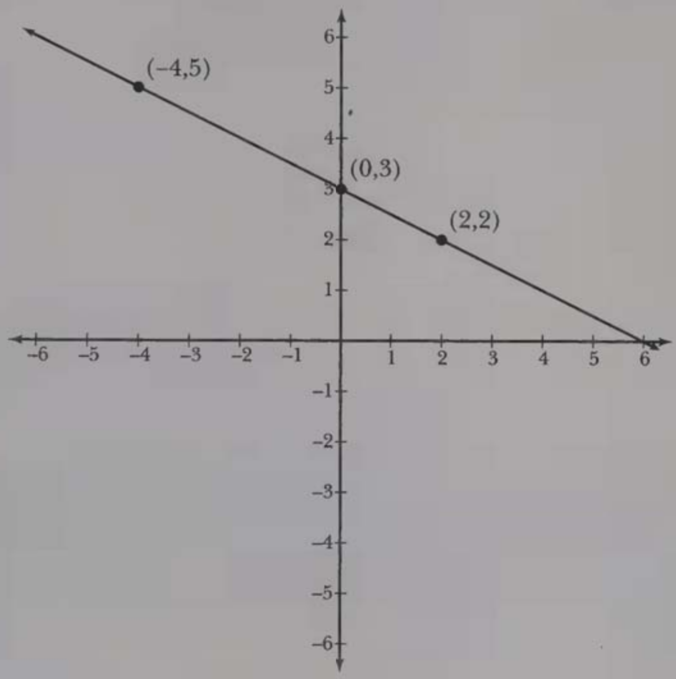
Choosing $x = 0$ for your table is usually a good idea, because any term containing an x will vanish, making your calculations easy.

x	$y=-\frac{1}{2}x+3$
−4	$y=-\frac{1}{2}(-4)+3$ $=\frac{4}{2}+3$ $=2+3$ $=5$
0	$y=-\frac{1}{2}(0)+3$ $=0+3$ $=3$
2	$y=-\frac{1}{2}(2)+3$ $=-\frac{2}{2}+3$ $=-1+3$ $=2$

Note: In Problems 18.25–18.26, you create a table of values to graph linear equation $y = -\frac{1}{2}x + 3$.

18.26 Construct points using the pairs of x- and y-values you generated in Problem 18.25 and plot them on a coordinate plane to graph the linear equation.

According to Problem 18.25, substituting $x = -4$, $x = 0$, and $x = 2$ into the equation produces the following y-values, respectively: $y = 5$, $y = 3$, and $y = 2$. Thus, the graph of the line passes through the points (–4,5), (0,3), and (2,2).



Note: In Problems 18.27–18.28, you create a table of values to graph linear equation $2(-3x + y) - 1 = y - 2x - 6$.

18.27 Choose three different values of x and substitute them into the equation to create a table of values.

Before you select values of x to substitute into the equation, you should solve the equation for y. Otherwise, you will need to solve for y each time you choose an x-value. Notice that the equations in Problems 18.23 and 18.25 are already solved for y.

$$2(-3x + y) - 1 = y - 2x - 6$$
$$2(-3x) + 2(y) - 1 = y - 2x - 6$$
$$-6x + 2y - 1 = y - 2x - 6$$

Isolate the y-term on the left side of the equal sign by subtracting y, adding $6x$, and adding 1 to both sides of the equation.

$$\begin{aligned}(-6x+6x)+(2y-y)+(-1+1)&=(y-y)+(-2x+6x)+(-6+1)\\0+y+0&=0+4x-5\\y&=4x-5\end{aligned}$$

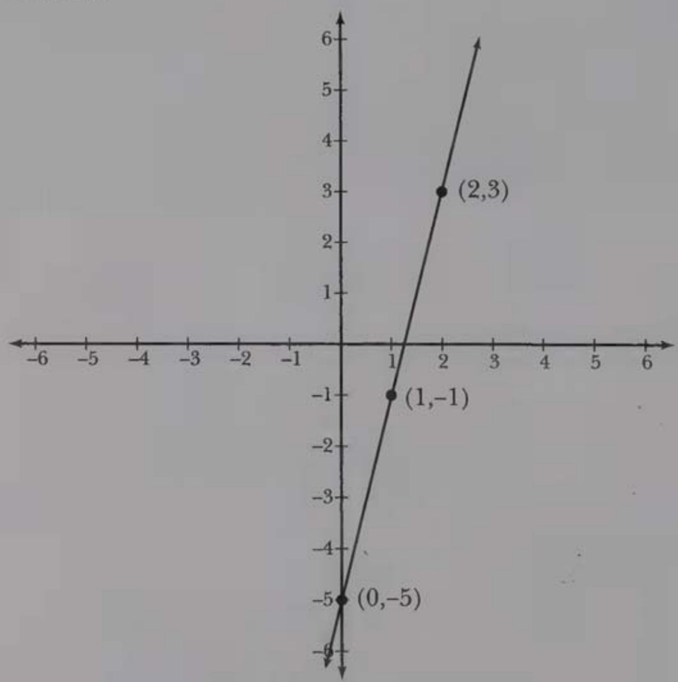
Choose three values of x to substitute into the equation $y = 4x - 5$ and create a table of values. The solution below uses the x-values 0, 1, and 2, but you may select any real number values for x.

x	$y=4x-5$
0	$y=4(0)-5$ $=0-5$ $=-5$
1	$y=4(1)-5$ $=4-5$ $=-1$
2	$y=4(2)-5$ $=8-5$ $=3$

Note: In Problems 18.27–18.28, you create a table of values to graph linear equation $2(-3x + y) - 1 = y - 2x - 6$.

18.28 Use the table of values generated in Problem 18.27 to graph the equation.

Use the x-coordinates you chose in Problem 18.27 and the corresponding y-coordinates generated by the table of values to create three coordinate pairs. According to Problem 18.27, the graph of the line passes through points (0,–5), (1,–1), and (2,3).



18.29 Create a table of values and use it to graph the linear equation $x - 3y = 15$.

Like the equation in Problem 18.28, this equation should be solved for y before you create a table of values.

$$\begin{aligned} x - 3y &= 15 \\ (x - x) - 3y &= 15 + (-x) \\ -3y &= 15 - x \\ \frac{-3y}{-3} &= \frac{15 - x}{-3} \\ y &= \frac{15 - x}{-3} \end{aligned}$$

You don't HAVE to do this, but if you do, you see that the x-values you choose for the table should be divisible by 3 to avoid y's that are fractions.

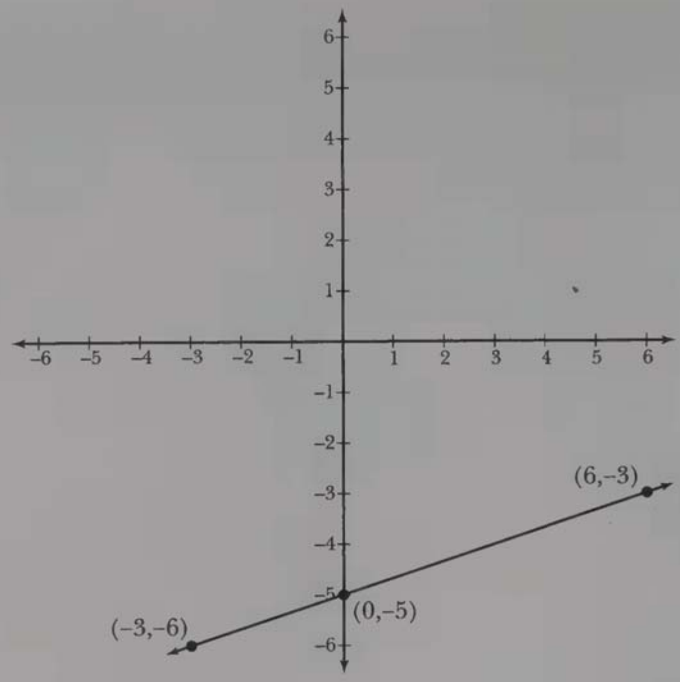
Divide each of the terms in the numerator by the denominator.

$$\begin{aligned} y &= \frac{15}{-3} + \frac{-x}{-3} \\ &= -5 + \frac{x}{3} \end{aligned}$$

Choose three x-values to construct a table of values. The solution below substitutes $x = -3$, $x = 0$, and $x = 6$ into the equation.

x	$y = -5 + \frac{x}{3}$
-3	$y = -5 + \frac{-3}{3}$ $= -5 + (-1)$ $= -5 - 1$ $= -6$
0	$y = -5 + \frac{0}{3}$ $= -5 + 0$ $= -5$
6	$y = -5 + \frac{6}{3}$ $= -5 + 2$ $= -3$

Plot the resulting points on the coordinate plane. The above table of values generates points $(-3,-6)$, $(0,-5)$, and $(6,-3)$.

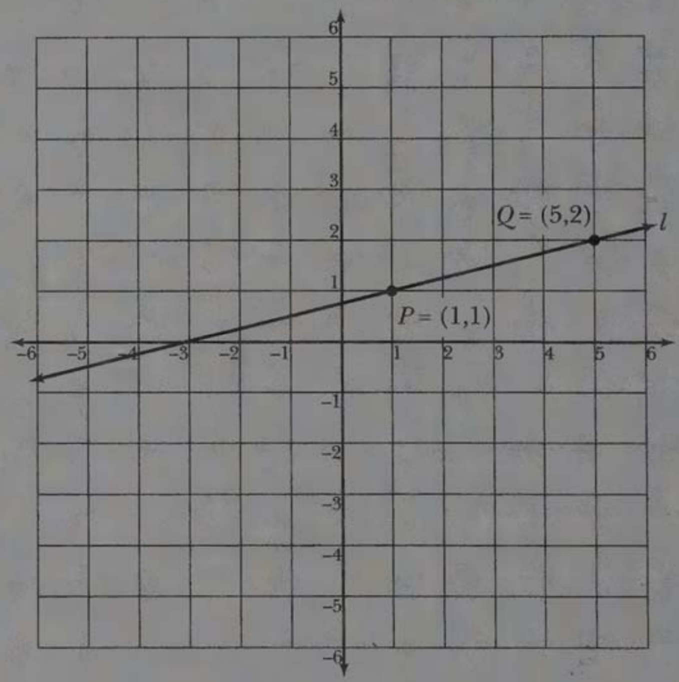


Calculating Slope of a Line

Solve for y or use $\frac{y_2 - y_1}{x_2 - x_1}$

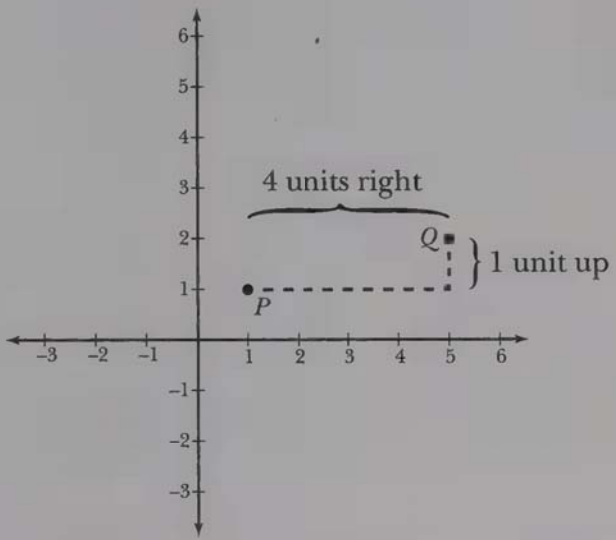
18.30 Line l passes through points $P = (1,1)$ and $Q = (5,2)$ in the graph below. Identify the slope of the line by starting at P and counting the total horizontal and vertical distance you travel to arrive at Q.

Literally put your finger on point P and count the horizontal and vertical lines you cross until you get to point Q.



The slope of a line is defined as its rise (the distance you travel up or down) divided by its run (the distance you travel right or left), as you pass from one point on the line to another. This quotient is constant no matter which two points you choose on the line or the direction you choose to travel between those points.

In Problems 18.34–18.38, you will apply a formula to calculate the rise and run (the *change in y* and the *change in x*) between points on a line. For now, simply begin at one point and jump from gridline to gridline, counting the total distances you travel horizontally and vertically. This problem directs you to start at P and travel to Q. This requires you to travel a total of four units to the right and one unit up, as illustrated below.

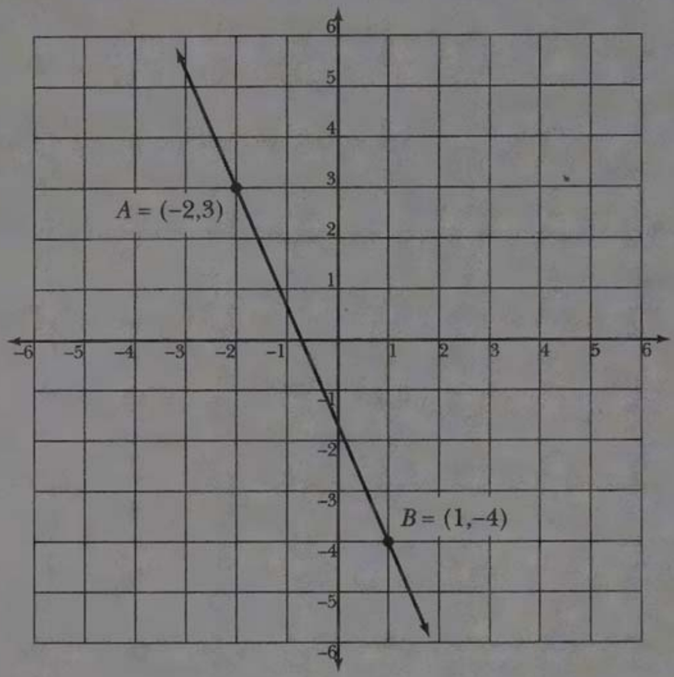


Remember that the slope is defined as the rise divided by the run, the change in y divided by the change in x. Thus, you should divide the vertical change (1) by the horizontal change (4). According to mathematics convention, the slope of a line is represented by the variable m.

$$\text{slope} = \frac{\text{change in } y}{\text{change in } x}$$
$$m = \frac{1}{4}$$

The slope of line l is $m = \frac{1}{4}$.

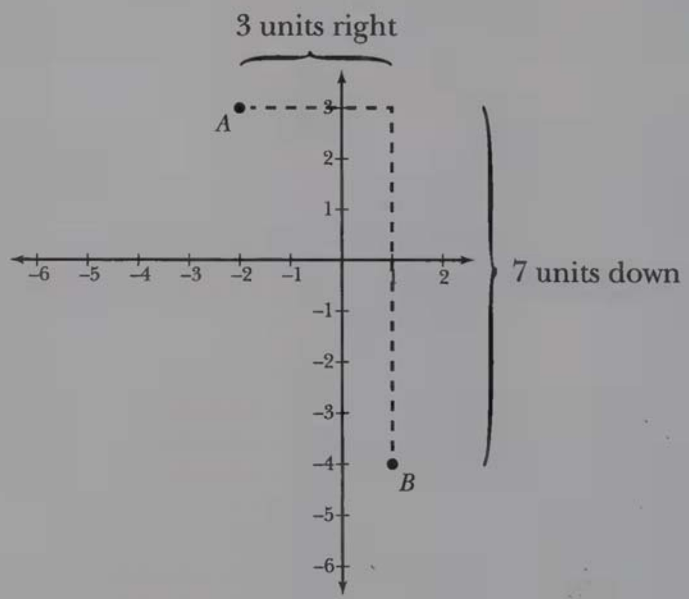
Note: In Problems 18.31–18.32, you travel between points A = (–2,3) and B = (1,–4) in two different directions to calculate the slope of the line illustrated in the graph below.



This works just like x- and y-values in the coordinate plane.
+x = move right
–x = move left
+y = move up
–y = move down

18.31 Identify the horizontal and vertical distances traveled when you begin at point A in Quadrant II and end at point B in Quadrant IV.

Like Problem 18.30, you should place your finger at point A and count the units as you jump from gridline to gridline to reach point B. You will travel a total of three units to the right and seven units down, as illustrated below. The direction in which you travel is important. Moving right is considered a positive change in x; moving down is a negative change in y.



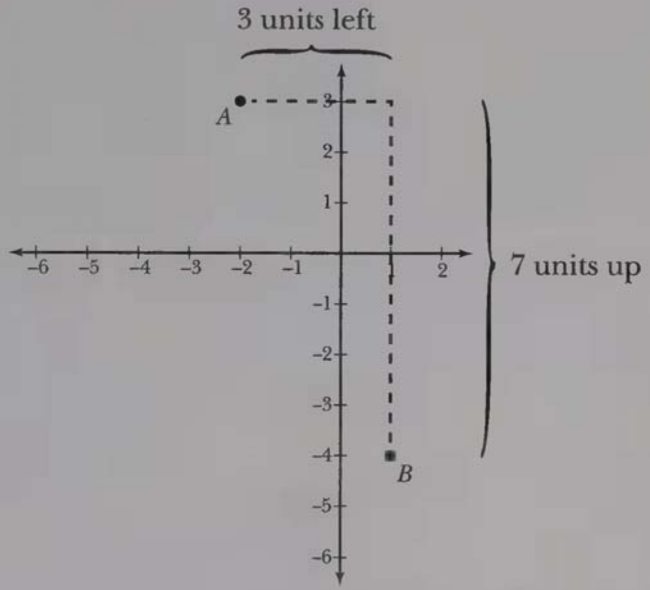
The slope (m) of the line passing through A and B is defined as the change in y (–7) divided by the change in x (+3).

$$m = \frac{-7}{+3}$$
$$m = -\frac{7}{3}$$

Note: In Problems 18.31–18.32, you travel between points A = (–2,3) and B = (1,–4) in two different directions to calculate the slope of the line illustrated in Problem 18.31.

18.32 Identify the horizontal and vertical distances traveled if you begin at point B in Quadrant IV and end at point A in Quadrant II.

Traveling from point B to point A requires you to travel up seven units and left three units, as illustrated below.

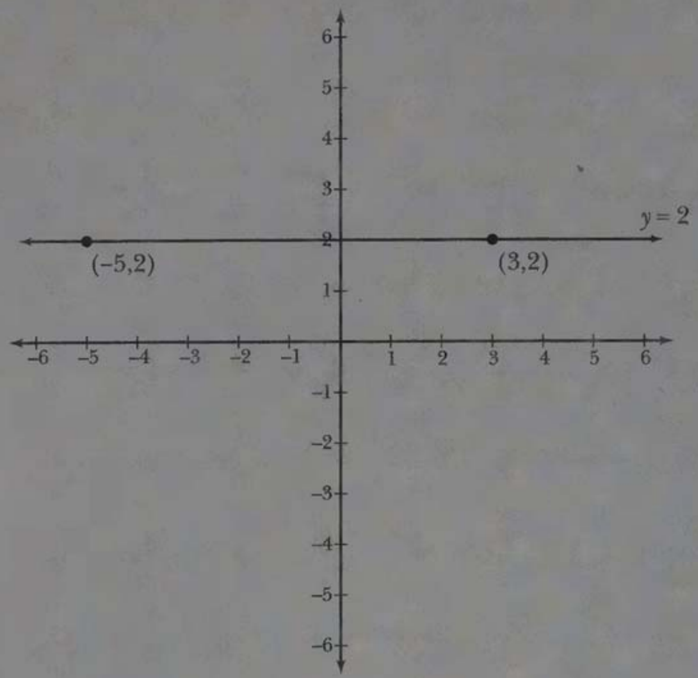


The total change in y is +7 and the total change in x is –3. Like traveling downward in Problem 18.31 resulted in a negative change in y, traveling leftward in this problem results in a negative change in x.

$$m = \frac{+7}{-3}$$
$$m = -\frac{7}{3}$$

Notice that the slope exactly matches the slope calculated in Problem 18.31, so no matter which direction you travel along the line, you get the same result.

18.33 Use the graph of horizontal line $y = 2$ below, and the two points identified on the graph, to calculate the slope of the line.



To travel from (–5,2) to (3,2), you move eight units to the right, so the total change in x is +8. (As Problems 18.31–18.32 demonstrate, you could also move in the opposite direction and will produce the same slope once the calculations are complete.) However, you do not move up or down along the path—both points are on the same horizontal line so there is zero change in y. Recall that the slope is the quotient of the change in y and the change in x.

$$\text{slope} = \frac{\text{change in } y}{\text{change in } x}$$
$$m = \frac{0}{8}$$
$$m = 0$$

Zero divided by any non-zero number is equal to zero. Therefore, the slope of the line $y = 2$ is $m = 0$.

ALL horizontal lines have a slope of zero, not just this particular horizontal line.

18.34 Verify your solution to Problem 18.30 by applying the slope formula to calculate the slope of the line passing through points (1,1) and (5,2).

In order to calculate the slope of a line passing through two points without graphing those points and counting the distances you travel on the coordinate plane, you can apply the slope formula. It states that a line passing through points (x_1, y_1) and (x_2, y_2) has slope $m = \frac{y_2 - y_1}{x_2 - x_1}$.

In other words, the slope is equal to the difference of the y-values divided by the difference of the x-values. However, it is important to note that the differences must be calculated in the same order. For example, if you subtract the y-coordinate of the first point from the y-coordinate of the second point in the numerator, then you *must* subtract the x-coordinate of the first point from the x-coordinate of the second point in the denominator as well.

In this problem, a line passes through points (1,1) and (5,2), so $x_1 = 1$, $y_1 = 1$, $x_2 = 5$, and $y_2 = 2$. Substitute all four of the coordinates into the slope formula and simplify.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{2-1}{5-1} \\ &= \frac{1}{4} \end{aligned}$$

According to Problem 18.30, the slope of the line is $m = \frac{1}{4}$, so you have successfully verified the answer.

Note that the slope formula does nothing more than calculate the distances that you calculated by hand in Problems 18.30–18.33. The numerator of the formula $(2 - 1 = 1)$ states that you move a total of one unit up as you travel from (1,1) to (5,2); the denominator $(5 - 1 = 4)$ states that you travel a total of four units to the right. However, the formula does not require you to graph the points, so it generally produces an accurate answer more quickly.

The formula is also much better than counting by hand when the points have fractions for coordinates.

18.35 Calculate the slope of the line passing through the points (–4,6) and (–1,–9).

Apply the slope formula introduced in Problem 18.34, setting $x_1 = -4$, $y_1 = 6$, $x_2 = -1$, and $y_2 = -9$.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-9-(6)}{-1-(-4)} \\ &= \frac{-9-6}{-1+4} \\ &= \frac{-15}{3} \\ &= -5 \end{aligned}$$

The slope of the line passing through points (–4,6) and (–1,–9) is $m = -5$.

18.36 Calculate the slope of the vertical line $x = -1$ by selecting two points on the line and applying the slope formula.

A vertical line doesn't have slope at all. It just sticks straight up into the air. Horizontal lines are perfectly level, so they have a slope of zero. Undefined slopes and zero slopes are not the same thing—they're opposites.

Points on the vertical line $x = -1$ have an x-coordinate of -1 and a y-coordinate that is any real number. For example, the points $(-1, 3)$ and $(-1, -10)$ lie on the vertical line; these points are used in the solution below. However, you may choose any two points $(-1, y_1)$ and $(-1, y_2)$, as long as both points have x-coordinate -1.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-10-(3)}{-1-(-1)} \\ &= \frac{-10-3}{-1+1} \\ &= \frac{-13}{0} \end{aligned}$$

Arithmetic rules prohibit division by zero; the result is best described as "undefined." Therefore, the slope of vertical line $x = -1$ is undefined. In fact, all vertical lines have undefined slopes.

18.37 A line k passes through points $(3,-5)$ and $(-7, c)$, where c is a real number. For what value of c is the slope of k equal to $\frac{3}{5}$?

In this problem, the y-coordinate of the second point is not known. However, you can calculate it by substituting $x_1 = 3$, $y_1 = -5$, $x_2 = -7$, $y_2 = c$, and $m = \frac{3}{5}$ into the slope formula and solving for c.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ \frac{3}{5} &= \frac{c-(-5)}{-7-(3)} \\ \frac{3}{5} &= \frac{c+5}{-7-3} \\ \frac{3}{5} &= \frac{c+5}{-10} \end{aligned}$$

Cross-multiply to solve the proportion. If you need to review proportions and cross-multiplication, see Problems 12.26–12.27.

$$3(-10) = 5(c+5)$$

Apply the distributive property and solve for c.

$$\begin{aligned} 3(-10) &= 5(c)+5(5) \\ -30 &= 5c+25 \\ (-30-25) &= 5c+(25-25) \\ -55 &= 5c \\ \frac{-55}{5} &= \frac{5c}{5} \\ -11 &= c \end{aligned}$$

If $c = -11$, then line k has slope $\frac{3}{5}$.

18.38 Calculate the slope of the line passing through the points $\left(\frac{1}{2},-\frac{5}{4}\right)$ and $\left(-\frac{15}{4},\frac{13}{8}\right)$.

Substitute $x_1 = \frac{1}{2}$, $y_1 = -\frac{5}{4}$, $x_2 = -\frac{15}{4}$, and $y_2 = \frac{13}{8}$ into the slope formula.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{\frac{13}{8} - \left(-\frac{5}{4}\right)}{-\frac{15}{4} - \left(\frac{1}{2}\right)}$$

$$m = \frac{\frac{13}{8} + \frac{5}{4}}{-\frac{15}{4} - \frac{1}{2}}$$

Least common denominators are required to combine the fractions in the numerator and denominator.

$$m = \frac{\frac{13}{8} + \frac{5}{4}\cdot\frac{2}{2}}{-\frac{15}{4} - \frac{1}{2}\cdot\frac{2}{2}}$$

$$m = \frac{\frac{13}{8} + \frac{10}{8}}{-\frac{15}{4} - \frac{2}{4}}$$

$$m = \frac{23/8}{-17/4}$$

Start by multiplying the numerator by the reciprocal of the denominator. (See Problem 9.31.)

Simplify the complex fraction.

$$m=\frac{23}{8}\left(-\frac{4}{17}\right)$$

Numerator value 4 and denominator value 8 have greatest common factor 4. Simplify the product before you multiply.

$$m=\frac{23}{2\,\cancel{8}}\left(-\frac{1\,\cancel{4}}{17}\right)$$

$$m=\frac{23}{2}\left(-\frac{1}{17}\right)$$

$$m=-\frac{23}{34}$$

The slope of the line passing through points $\left(\frac{1}{2},-\frac{5}{4}\right)$ and $\left(-\frac{15}{4},\frac{13}{8}\right)$ is $m=-\frac{23}{34}$.

18.39 Solve the linear equation $4x+y=7$ for y to calculate the slope of its graph.

Problems 18.30–18.33 demonstrate how to calculate the slope of a line by counting the horizontal and vertical distances between two points on the graph of the line. Problems 18.34–18.38 extend that concept, replacing the physical act of counting distances on a graph with a formula that accomplishes the same goal. In the final three problems of this section, you are presented with a third method for calculating the slope of a line.

If you solve a linear equation for y, such that y is on one side of the equation and exactly two terms lie on the other side of the equation (one x-term and one constant), the equation is said to be in *slope-intercept form*. The coefficient of the x-term is the slope of the line and the constant is the location at which the line intersects the y-axis. For now, you will focus on calculating the slope of the line.

To solve $4x+y=7$ for y, subtract $4x$ from both sides of the equation.

$$y=-4x+7$$

The equation is in slope-intercept form, because it is solved for y and the other side of the equation contains exactly two terms, an x-term and a constant. Therefore, the slope of the line is $m=-4$, the coefficient of x.

18.40 Solve the linear equation $3x-2y=18$ for y to calculate the slope of its graph.

Solve the equation for y by subtracting $3x$ from both sides and then dividing each of the terms in the equation by -2, the coefficient of y.

$$-2y=-3x+18$$

$$\frac{-2y}{-2}=\frac{-3x}{-2}+\frac{18}{-2}$$

$$y=\frac{3}{2}x-9$$

The coefficient of the x-term is the slope of the line: $m=\frac{3}{2}$.

18.41 Solve the linear equation $2(5x-3y)+9=8x+1$ for y to calculate the slope of its graph.

Simplify the left side of the equation by applying the distributive property.

$$\begin{aligned} 2(5x-3y)+9&=8x+1\\ 2(5x)+2(-3y)+9&=8x+1\\ 10x-6y+9&=8x+1 \end{aligned}$$

Isolate y by subtracting $10x$ and 9 from both sides of the equation and then dividing each term by -6, the coefficient of y.

$$\begin{aligned} (10x-10x)-6y+(9-9)&=(8x-10x)+(1-9)\\ -6y&=-2x-8\\ \frac{-6y}{-6}&=\frac{-2x}{-6}+\frac{-8}{-6}\\ y&=\frac{2}{6}x+\frac{8}{6} \end{aligned}$$

Reduce the fractions to lowest terms.

$$\begin{aligned} y&=\frac{2\div 2}{6\div 2}x+\frac{8\div 2}{6\div 2}\\ y&=\frac{1}{3}x+\frac{4}{3} \end{aligned}$$

The coefficient of the x-term is the slope of the line: $m=\frac{1}{3}$.

Chapter 19
BASIC GEOMETRY

Important facts about simple shapes

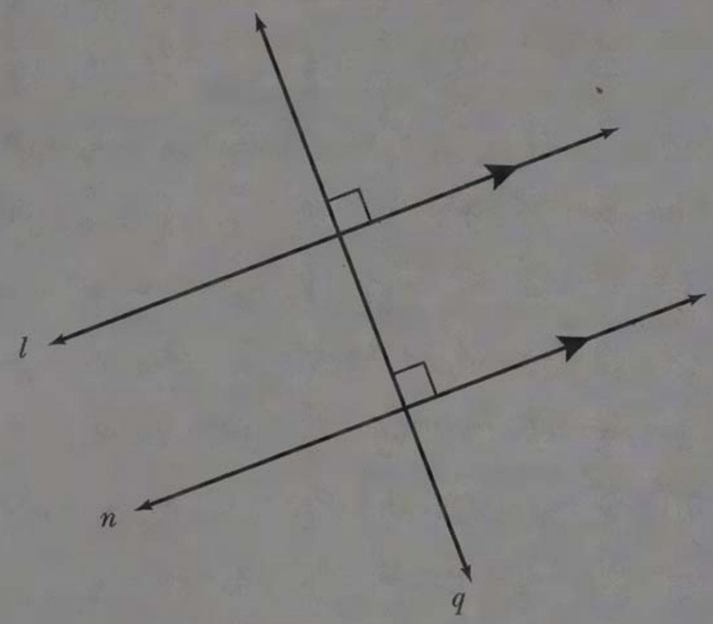
The majority of this book is directed toward the introduction and exploration of algebraic concepts and algorithms, but the final three chapters present cursory introductions to other basic mathematics topics. This chapter discusses fundamental geometric concepts, focusing primarily on two- and three-dimensional figures and shapes. As you will learn in subsequent courses, geometry is more than simply measuring and quantifying these objects. Logical thinking and proof are integral parts of geometry, but they are sufficiently complex topics to require more space than may be allotted in this chapter.

You're only going to spend one chapter on geometry, so there's not enough time or space to go into great detail. However, there are enough shapes and figures in here to keep you busy, and each of them has its own particular area, perimeter, and/or volume formulas to learn. They're all listed in Appendix B: Important Formulas for easy access. By the way, you'll see a handful of the algebra topics covered in preceding chapters sprinkled throughout, including square roots.

Lines and Angles

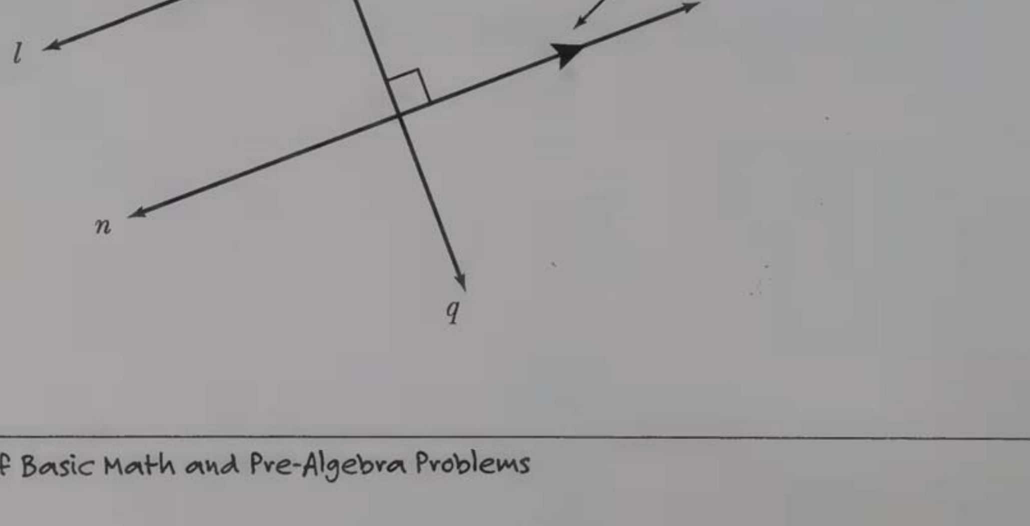
Parallel, perpendicular, acute, obtuse, and right

Note: Problems 19.1–19.3 refer to the diagram below.



19.1 Identify the parallel lines in the diagram and explain your answer.

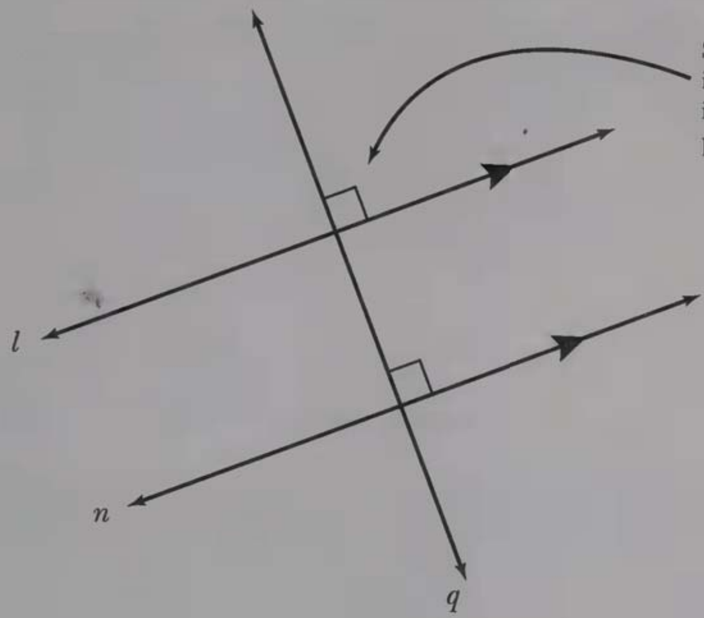
Parallel lines have equal slopes, so they stay the same distance apart from each other infinitely in each direction, never intersecting. This diagram contains parallel lines l and n. In order to assure students that lines *appearing* parallel truly are, textbooks often indicate parallel lines using arrowheads that appear on the lines themselves, in addition to the arrowheads placed at the ends of the lines.



Note: Problems 19.1–19.3 refer to the diagram in Problem 19.1.

19.2 Identify the perpendicular lines in the diagram and explain your answer.

Perpendicular lines meet at right angles—angles that measure 90 degrees. In this diagram, lines *l* and *n* are both perpendicular to line *q*. If you are not familiar with the concept of a right angle, picture a square, whose four sides all intersect in right angles. In fact, small squares are often used in geometric diagrams to indicate perpendicular lines.



Small squares placed at the intersection of two lines indicate that those lines are perpendicular to each other.

Note: Problems 19.1–19.3 refer to the diagram in Problem 19.1.

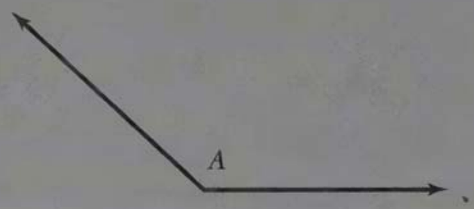
19.3 Use the diagram to complete the following statement and justify your answer.

Two lines __________ to the same line are __________ to each other.

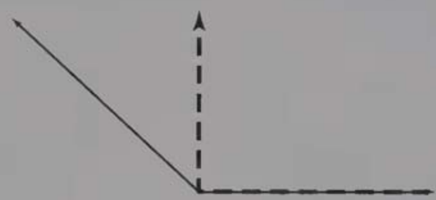
In this diagram, lines *l* and *n* are both perpendicular to the same line, *q*. In addition, lines *l* and *n* are parallel to each other. Thus, *two lines* <u>*perpendicular*</u> *to the same line are* <u>*parallel*</u> *to each other.* The statement is not only true for this specific diagram. Rather, it is true in general, and proving this statement is a common exercise for a new student in a geometry course.

To learn how to prove this statement, look at Problem 6.23 in The Humongous Book of GEOMETRY Problems.

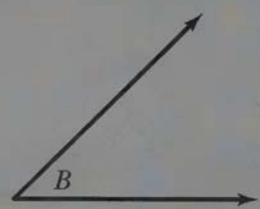
19.4 Determine whether angle A below is acute or obtuse and explain your answer.



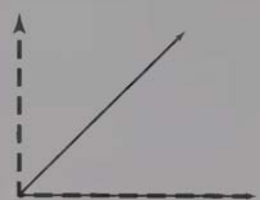
Angle A is an *obtuse* angle because its measure is larger than 90°; note that the small, circular symbol "°" represents *degrees*. In other words, angle A is larger than a right angle, which measures exactly 90°. The diagram below verifies this assumption visually, by constructing a pair of dotted perpendicular lines. The right sides of the right angle and angle A overlap, but the left side of angle A extends beyond the left side of the dotted right angle.



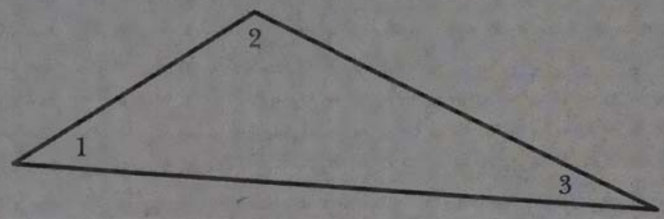
19.5 Determine whether angle B below is acute or obtuse and explain your answer.



As the diagram below demonstrates, angle B is smaller than a right angle, and thus measures less than 90°. Any angle measuring between (but not including) 0° and 90° is defined as an *acute* angle. Thus, angle B is acute.

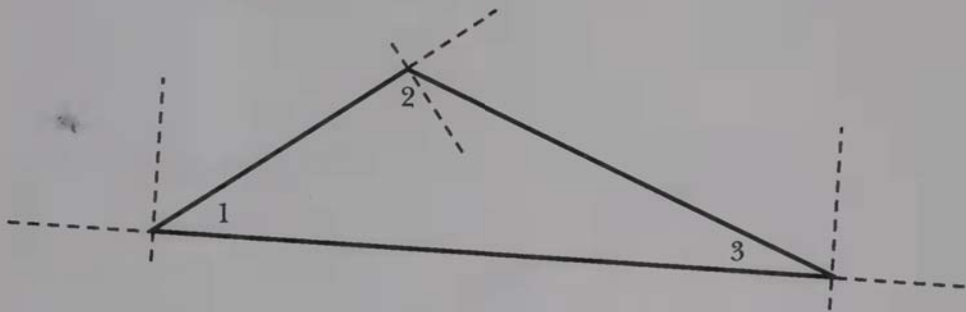


19.6 Classify each of the numbered angles in the figure below as acute or obtuse and explain your answers.



Extend the sides of the triangle beyond the points at which the sides intersect and then draw lines perpendicular at each vertex of the triangle. In the diagram below, the extended sides and the lines perpendicular to those sides are drawn as dotted segments.

The vertex is the "corner" of a shape, where two sides intersect.



Only angle 2 is larger than a right angle, so angles 1 and 3 are acute, and angle 2 is obtuse.

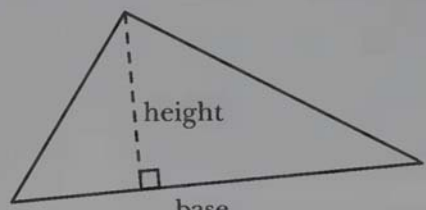
Triangles

Area, perimeter, and the Pythagorean Theorem

19.7 Calculate the area of a triangle with base 10 and height 6.

The area A of a triangle is defined as half the product of its base b and height h; thus, $A = \frac{1}{2}bh$. Note that you do not necessarily need to know the lengths of all three sides of the triangle. Rather, you need the length of a single side—any side—called the base, and the corresponding height, the length of the segment perpendicular to the base that ends at the opposite vertex of the triangle.

Consider the triangle below, in which one side of the triangle is identified as the base. The dotted segment in the diagram is perpendicular to the base and extends to the vertex of the triangle opposite that base. The length of the dotted line is the height of the triangle.



In this problem, you are given $b = 10$ and $h = 6$. Substitute these values into the triangle area formula and simplify the expression.

$$\begin{aligned} A &= \frac{1}{2}bh \\ &= \frac{1}{2}(10)(6) \end{aligned}$$

Multiply from left to right, beginning by calculating half of 10, and then multiplying the result by 6.

$$\begin{aligned} &= 5(6) \\ &= 30 \end{aligned}$$

The area of the triangle with base 10 and height 6 is 30.

19.8 Calculate the area of a triangle with base 8 and height $\frac{7}{6}$.

Substitute $b = 8$ and $h = \frac{7}{6}$ into the triangle area formula $A = \frac{1}{2}bh$.

$$\begin{aligned} A &= \frac{1}{2}bh \\ &= \frac{1}{2}(8)\left(\frac{7}{6}\right) \\ &= \frac{1}{2}\left(\frac{8}{1}\right)\left(\frac{7}{6}\right) \end{aligned}$$

Multiply from left to right.

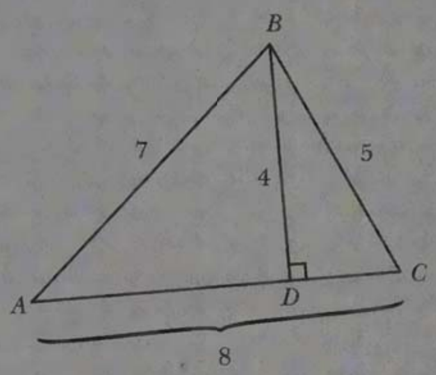
$$\begin{aligned} &= \left(\frac{1 \cdot 8}{2 \cdot 1}\right)\left(\frac{7}{6}\right) \\ &= \left(\frac{8}{2}\right)\left(\frac{7}{6}\right) \\ &= \left(\frac{8 \div 2}{2 \div 2}\right)\left(\frac{7}{6}\right) \\ &= \left(\frac{4}{1}\right)\left(\frac{7}{6}\right) \end{aligned}$$

The numerator value 4 and the denominator value 6 share greatest common factor 2. Reduce the product before you multiply.

$$=\left(\frac{2\,\cancel{4}}{1}\right)\left(\frac{7}{3\,\cancel{6}}\right)$$
$$=\left(\frac{2}{1}\right)\left(\frac{7}{3}\right)$$
$$=\frac{14}{3}$$

The area of the triangle with base 8 and height $\frac{7}{6}$ is $\frac{14}{3}$.

Note: Problems 19.9–19.10 refer to triangle ABC illustrated below.



19.9 Calculate the perimeter of triangle ABC.

The sides of triangle ABC have lengths $AB = 7$, $BC = 5$, and $AC = 8$. The perimeter of the figure is defined as the sum of the lengths of the sides.

$$\begin{aligned}\text{perimeter of triangle } ABC &= AB + BC + AC\\ &= 7+5+8\\ &= 12+8\\ &= 20\end{aligned}$$

In geometry, you identify the lengths of sides using the two letters that mark the endpoints of the sides. For example, the left side of this triangle has endpoints A and B, so you can say AB = 7 or BA = 7. Usually, the letters are written in alphabetical order.

Note: Problems 19.9–19.10 refer to triangle ABC, illustrated in Problem 19.9.

19.10 Calculate the area of triangle ABC.

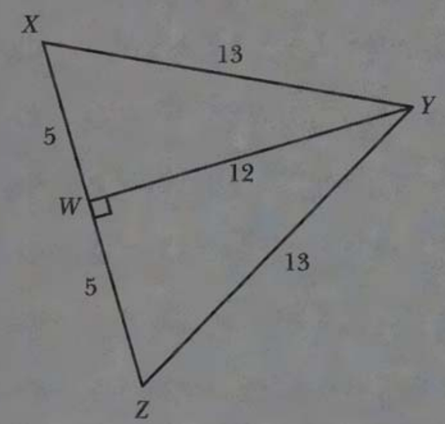
As Problem 19.7 explains, the height of a triangle is defined as the length of the segment that is perpendicular to one side of the triangle (called the base) and intersects the opposite vertex. In this diagram, the segment with endpoints B and D is perpendicular to the side of triangle ABC with endpoints A and C. (This is written $\overline{BD} \perp \overline{AC}$; the symbol "⊥" is read "is perpendicular to," and the segments are named by their endpoints.) In addition, segment $\overline{BD}$ intersects point B, the vertex opposite side $\overline{AC}$.

Just like the symbol for "perpendicular" looks like perpendicular lines, the symbol for "parallel" looks like parallel lines: ||.

Therefore, if the base of the triangle has length $AC = 8$, the corresponding height is $BD = 4$. Substitute these values into the triangle area formula.

$$\begin{aligned} A &= \frac{1}{2}bh \\ &= \frac{1}{2}(8)(4) \\ &= \left(\frac{8}{2}\right)(4) \\ &= 4(4) \\ &= 16 \end{aligned}$$

19.11 Calculate the area and perimeter of triangle XYZ below.



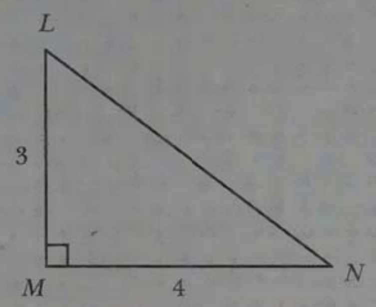
The perimeter of a triangle is the sum of the lengths of its sides. Notice that two sides of triangle XYZ have the same length. Any triangle in which at least two of the sides have the same length is *isosceles*. Notice that you must add lengths XW and WZ in order to calculate XZ: $5 + 5 = 10$.

$$\begin{aligned} \text{perimeter of } XYZ &= XY + YZ + XZ \\ &= 13 + 13 + 10 \\ &= 26 + 10 \\ &= 36 \end{aligned}$$

Notice that $\overline{XZ} \perp \overline{WY}$, so the base with length $XZ = 10$ has a corresponding height of $WY = 12$. Substitute these values into the triangle area formula.

$$\begin{aligned} A &= \frac{1}{2}bh \\ &= \frac{1}{2}(10)(12) \\ &= \left(\frac{10}{2}\right)(12) \\ &= 5(12) \\ &= 60 \end{aligned}$$

19.12 Calculate the length of the hypotenuse in right triangle *LMN* below.



If two sides of a triangle are perpendicular, then those sides are defined as the *legs* of the *right triangle*. The lengths of the sides of a right triangle satisfy the following formula, called the Pythagorean Theorem. Assume that a and b represent the lengths of the legs and c represents the length of the hypotenuse.

The legs are perpendicular, so they meet at a right angle.

$$a^2 + b^2 = c^2$$

Note that the hypotenuse of a right triangle is opposite the right angle, and it is always the longest side: $c > a$ and $c > b$. In triangle *LMN*, the perpendicular sides—the legs of the right triangle—have lengths 3 and 4. Thus, you can substitute $a = 3$ and $b = 4$ into the Pythagorean Theorem to solve for c. Note that substituting $a = 4$ and $b = 3$ into the formula produces the same final result; it does not matter which leg length you substitute into each variable on the left side of the Pythagorean Theorem.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ (3)^2 + (4)^2 &= c^2 \\ 9 + 16 &= c^2 \\ 25 &= c^2 \end{aligned}$$

To solve the equation $25 = c^2$, take the square root of both sides of the equation. Because an exponent of two and a root of two—a square root—are opposites, the operations cancel one another, and the result is $\sqrt{c^2} = c$.

If you square c then take the square root, you get c.

$$\begin{aligned} \sqrt{25} &= \sqrt{c^2} \\ 5 &= c \end{aligned}$$

The length of the hypotenuse of triangle *LMN* is $LN = 5$.

19.13 Calculate the length of the hypotenuse of a right triangle if its legs have lengths 8 and 15.

Substitute $a = 8$ and $b = 15$ into the Pythagorean Theorem and solve for the length of the hypotenuse, c.

$$\begin{aligned}
a^2 + b^2 &= c^2 \\
(8)^2 + (15)^2 &= c^2 \\
64 + 225 &= c^2 \\
289 &= c^2 \\
\sqrt{289} &= \sqrt{c^2} \\
17 &= c
\end{aligned}$$

The length of the hypotenuse of the right triangle is 17.

19.14 Calculate the area and perimeter of a right triangle with hypotenuse length 25 and a leg with length 24.

In this problem, you are given the length of the hypotenuse and one leg. Set $c = 25$ and either a or b equal to 24 in the Pythagorean Theorem—it does not matter which. Solve for the length of the remaining leg.

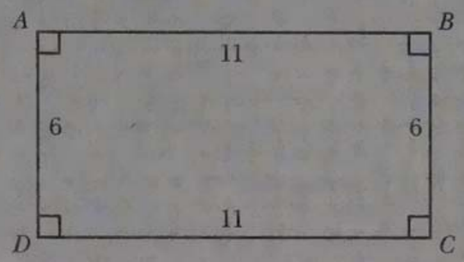
$$\begin{aligned}
a^2 + b^2 &= c^2 \\
(24)^2 + b^2 &= (25)^2 \\
576 + b^2 &= 625 \\
b^2 &= 625 - 576 \\
b^2 &= 49 \\
\sqrt{b^2} &= \sqrt{49} \\
b &= 7
\end{aligned}$$

If a right triangle has a hypotenuse of length 25 and a leg of length 24, the remaining leg has length 7.

Quadrilaterals

Rectangles, rhombi, squares, parallelograms, trapezoids

19.15 Calculate the area and perimeter of rectangle *ABCD* below.



The perimeter of a rectangle, like the perimeter of a triangle or any polygon, is the sum of the lengths of its sides. The sides of rectangle *ABCD* have lengths 11, 6, 11, and 6.

A polygon is a two-dimensional shape with sides that are line segments.

$$\begin{aligned}\text{perimeter of } ABCD &= AB + BC + CD + AD\\ &= 11+6+11+6\\ &= 34\end{aligned}$$

Notice that opposite sides of a rectangle have equal lengths. In this diagram, $AB = CD = 11$ and $AD = BC = 6$. The longer dimension is the *length* of the rectangle, and the shorter dimension is the *width*. Therefore, the length of rectangle *ABCD* is $l = 11$ and its width is $w = 6$.

You can also use the formula $P = 2l + 2w$ to calculate the perimeter of a rectangle, because two of the sides have length l and two of the sides have length w. This formula returns the same value for the perimeter of *ABCD* as calculated above.

$$\begin{aligned}P &= 2l+2w\\ &= 2(11)+2(6)\\ &= 22+12\\ &= 34\end{aligned}$$

The area of a rectangle is equal to the product of its length and width. Substitute $l = 11$ and $w = 6$ into the rectangle area formula $A = lw$.

$$\begin{aligned}A &= lw\\ &= (11)(6)\\ &= 66\end{aligned}$$

19.16 Calculate the perimeter and area of a rectangle with length $l = 9$ and width $w = 4$.

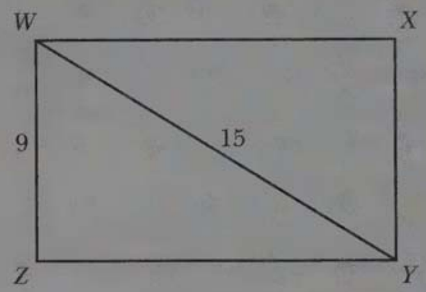
Apply the formulas presented in Problem 19.15. Recall that the perimeter is equal to twice the length plus twice the width of the rectangle.

$$\begin{aligned} P &= 2l + 2w \\ &= 2(9) + 2(4) \\ &= 18 + 8 \\ &= 26 \end{aligned}$$

The area of a rectangle is equal to the product of its length and width.

$$\begin{aligned} A &= lw \\ &= (9)(4) \\ &= 36 \end{aligned}$$

19.17 Calculate the area and perimeter of rectangle $WXYZ$, illustrated below.



A diagonal connects two corners of a figure but isn't a side of the figure—it slices through the interior. This diagonal literally cuts the rectangle in half.

You are given the length of one side ($WZ = 9$) and a diagonal ($WY = 15$). The sides of a rectangle intersect at right angles, so triangle WZY is a right triangle. Apply the Pythagorean Theorem to calculate length YZ.

$$\begin{aligned} (WZ)^2 + (YZ)^2 &= (WY)^2 \\ (9)^2 + (YZ)^2 &= (15)^2 \\ 81 + (YZ)^2 &= 225 \\ (YZ)^2 &= 225 - 81 \\ (YZ)^2 &= 144 \\ \sqrt{(YZ)^2} &= \sqrt{144} \\ YZ &= 12 \end{aligned}$$

Opposite sides of a rectangle have equal lengths. Therefore, $WZ = XY = 9$ and $WX = YZ = 12$. The longer of the dimensions is the length and the shorter is the width of the rectangle: $l = 12$ and $w = 9$. Apply the perimeter formula presented in Problem 19.15.

$$\begin{aligned} P &= 2l + 2w \\ &= 2(12) + 2(9) \\ &= 24 + 18 \\ &= 42 \end{aligned}$$

Multiply the length and width of the rectangle to calculate the area.

$$\begin{aligned} A &= lw \\ &= (12)(9) \\ &= 108 \end{aligned}$$

Note: Problems 19.18–19.20 refer to a rhombus with side length 10 and height 8.

19.18 Calculate the perimeter of the rhombus.

All four sides of a rhombus have equal lengths—if one side has length 10, then the three remaining sides have length 10 as well. Therefore, to calculate the perimeter P of a rhombus, multiply the length of a single side, s, by 4.

$$P = 4s$$

In this problem $s = 10$. Substitute it into the rhombus perimeter formula.

$$\begin{aligned} P &= 4s \\ &= 4(10) \\ &= 40 \end{aligned}$$

Note: Problems 19.18–19.20 refer to a rhombus with side length 10 and height 8.

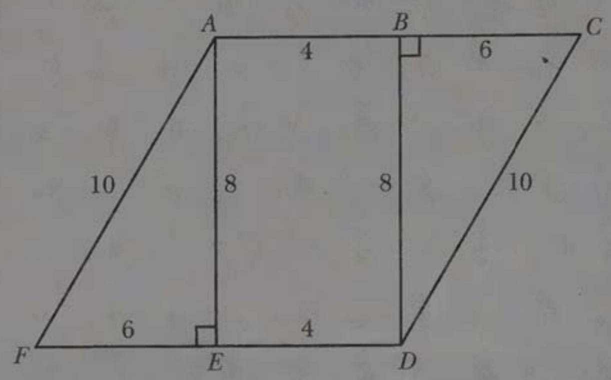
19.19 Calculate the area of a rhombus.

The height of a rhombus is the length of the perpendicular segment from a vertex to the side opposite it. This problem states that the height of the rhombus is $h = 8$. To calculate the area, multiply the side length $s = 10$ by the height.

$$\begin{aligned} A &= sh \\ &= (10)(8) \\ &= 80 \end{aligned}$$

Note: Problems 19.18–19.20 refer to a rhombus with side length 10 and height 8.

19.20 In the diagram below, the rhombus is divided into three figures, two right triangles and a rectangle. Add the areas of the figures to verify your answer to Problem 19.19.



The legs of a right triangle are perpendicular, so treat one of them as the base and the other as the height—it doesn't matter which is which.

Recall that the height of the rhombus, $AE = BD = 8$, is perpendicular to the side from which it extends. Therefore, angles AEF and DBC are right angles. Furthermore, triangles AEF and DBC are right triangles. Apply the triangle area formula presented in Problem 19.7, setting $b = 6$ and $h = 8$.

$$\begin{aligned} A_{\text{right triangle}} &= \frac{1}{2}bh \\ &= \frac{1}{2}(6)(8) \\ &= \left(\frac{6}{2}\right)(8) \\ &= 3(8) \\ &= 24 \end{aligned}$$

Each of the right triangles has area 24. Now calculate the area of rectangle $ABDE$, which has length 8 and width 4.

$$\begin{aligned} A_{\text{rectangle}} &= lw \\ &= 8(4) \\ &= 32 \end{aligned}$$

The area of rhombus $ACDF$ is equal to the sum of the areas of rectangle $ABDE$ and right triangles AEF and DBC.

$$\begin{aligned} \text{Area}(ACDF) &= \text{Area}(\square ABDE) + \text{Area}(\triangle AEF) + \text{Area}(\triangle DBC) \\ &= 32 + 24 + 24 \\ &= 80 \end{aligned}$$

The sum of the individual areas is equal to the area calculated in Problem 19.19 using the rhombus area formula.

19.21 Calculate the area of a rhombus with diagonals of length 4 and 6.

As Problem 19.19 explains, the area of a rhombus may be calculated by multiplying its height by the length of one side. In this problem, however, you are given neither length. Instead, the lengths of the diagonals are known. Apply the alternative rhombus area formula below, in which d_1 and d_2 represent the lengths of the diagonals and A is the area of the corresponding rhombus.

$$\begin{aligned} A &= \frac{1}{2}(d_1)(d_2) \\ &= \frac{1}{2}(4)(6) \\ &= \left(\frac{4}{2}\right)(6) \\ &= 2(6) \\ &= 12 \end{aligned}$$

19.22 Calculate the area and perimeter of a square with side length 7.

A square is both a rectangle and a rhombus. Like a rectangle, it contains four right angles, and like a rhombus, all four sides are the same length. Therefore, you can apply either the rectangle or rhombus area and perimeter formulas to calculate the area and perimeter of a square. You should choose the easier of the formulas in each case. For example, to calculate the perimeter of a square, apply the rhombus perimeter formula. Note that the square has side length $s = 7$.

$$\begin{aligned} P &= 4s \\ &= 4(7) \\ &= 28 \end{aligned}$$

To generate the area formula for a square, apply the rectangle area formula. Note that the length and width of a square are equal: $l = w = s$, given side length s. Substitute s into the rectangle area formula for both l and w.

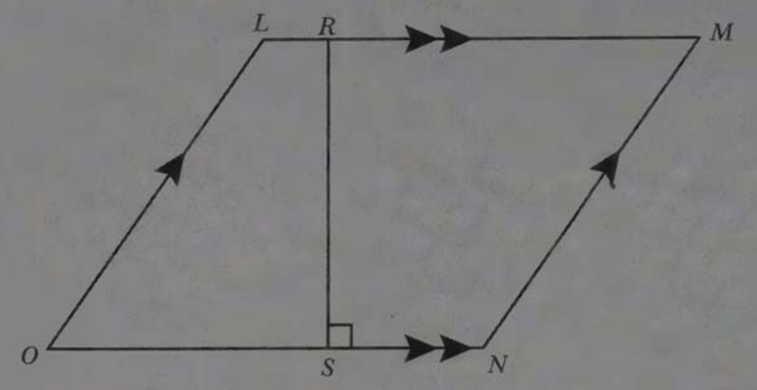
$$\begin{aligned} A &= lw \\ &= s(s) \\ &= s^2 \end{aligned}$$

The area of a SQUARE is the side length SQUARED!

Therefore, the area of a square is equal to the side length raised to the second power.

$$\begin{aligned} A &= s^2 \\ &= (7)^2 \\ &= 49 \end{aligned}$$

Note: Problems 19.23–19.24 refer to quadrilateral LMNO below, in which LM = 9, MN = 5, and RS = 3.



19.23 Classify *LMNO* as a specific type of quadrilateral and calculate its perimeter.

As explained in Problem 19.1, arrowheads in geometric diagrams indicate parallel lines. The left and right sides of this quadrilateral are parallel, as each has a single arrowhead on it. Similarly, $\overline{LM} \parallel \overline{ON}$ because those sides are illustrated with two arrowheads. Quadrilaterals with parallel opposite sides are *parallelograms.*

All of the quadrilaterals in the preceding problems of this section are also parallelograms, but they have additional characteristics that allow you to classify them as specific types of parallelograms. *LMNO* does not have those characteristics (its sides are not perpendicular nor are they all the same length), so it is not a rectangle, a rhombus, nor a square. However, because it is a parallelogram, you conclude that its opposite sides have the same length.

The opposite ANGLES of a parallelogram also have the same measure. In this figure, angles L and N are equal, as are M and O.

Therefore, $LM = ON = 9$ and $MN = LO = 5$. To calculate the perimeter of a parallelogram, add the lengths of its sides.

$$\begin{aligned}\text{Perimeter of } LMNO &= LM + MN + NO + LO \\ &= 9 + 5 + 9 + 5 \\ &= 28\end{aligned}$$

Note: Problems 19.23–19.24 refer to quadrilateral LMNO illustrated in Problem 19.23, in which LM = 9, MN = 5, and RS = 3.

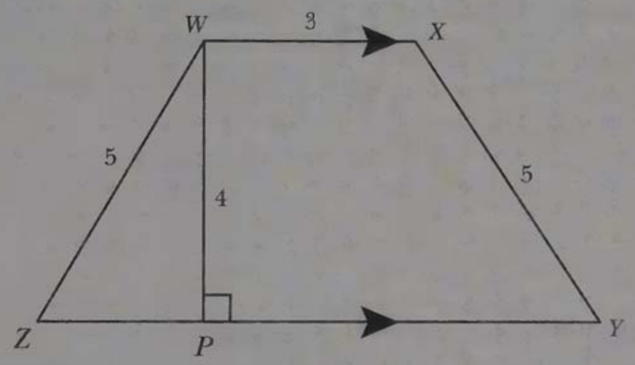
19.24 Calculate the area of *LMNO*.

To calculate the area of a parallelogram, multiply the length of one side (the *base*) by the perpendicular segment that stretches from a vertex to the opposite side of the figure (the *height h*): $A = bh$.

In parallelogram *LMNO*, $\overline{RS}$ is perpendicular to sides $\overline{LM}$ and $\overline{ON}$. Select one of those parallel sides as the base and set the height equal to the length of the perpendicular segment: $b = ON = 9$ and $h = RS = 3$. Substitute these values into the parallelogram area formula.

$$\begin{aligned} A &= bh \\ &= (9)(3) \\ &= 27 \end{aligned}$$

Note: Problems 19.25–19.26 refer to quadrilateral WXYZ below, in which $\overline{WP} \perp \overline{ZY}$ and YZ = 9.



19.25 Classify *WXYZ* as a specific type of quadrilateral and calculate its perimeter.

Unlike the quadrilateral in Problems 19.23–19.24, this quadrilateral has only one pair of parallel opposite sides; therefore, *WXYZ* is a *trapezoid*. The parallel sides of a trapezoid are called the *bases*, so the bases of *WXYZ* are $\overline{WX}$ and $\overline{YZ}$. The nonparallel sides of the trapezoid are the *legs*. The legs do not necessarily have the same length, but when they do—as they do in *WXYZ*—the figure is best classified as an *isosceles trapezoid*.

To compute the perimeter of a trapezoid, add the lengths of the bases and the legs. Note that you are given $YZ = 9$, the length of the longest side.

$$\begin{aligned} \text{Perimeter of } WXYZ &= WX + XY + YZ + WZ \\ &= 3 + 5 + 9 + 5 \\ &= 22 \end{aligned}$$

Note: Problems 19.25–19.26 refer to quadrilateral WXYZ illustrated in Problem 19.25, in which $\overline{WP} \perp \overline{ZY}$ and $YZ = 9$.

19.26 Calculate the area of $WXYZ$.

According to Problem 19.25, the parallel sides of a trapezoid are its bases. The area of a trapezoid is calculated according to the formula below, in which b_1 and b_2 are the lengths of the bases and h is the height of the trapezoid.

$$A = \frac{h}{2}(b_1 + b_2)$$

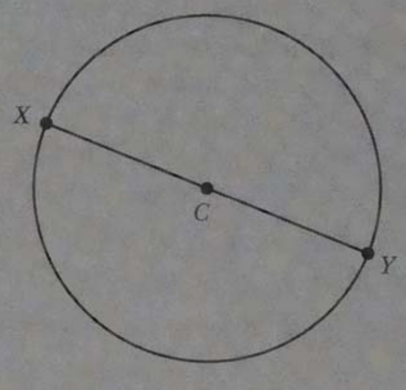
The height of a trapezoid is the length of the segment bounded by, and perpendicular to, the bases. The height of trapezoid $WXYZ$ is $WP = 4$. Substitute $b_1 = 3$, $b_2 = 9$, and $h = 4$ into the area formula. (Note that you can reverse the bases; substituting $b_1 = 9$ and $b_2 = 3$ into the formula produces the same result.)

$$\begin{aligned} A &= \frac{h}{2}(b_1 + b_2) \\ &= \frac{4}{2}(3+9) \\ &= \frac{4}{2}(12) \\ &= 2(12) \\ &= 24 \end{aligned}$$

Circles

Radius, diameter, area, circumference

Note: Problems 19.27–19.29 refer to circle C in the diagram below, with diameter 14 and points X and Y on the circle as illustrated.



19.27 Calculate the radius of circle C.

A circle is defined as the set of points that are equidistant from a fixed point, called the *center.* The center also names the circle. This diagram contains a circle named C, so the center of the circle is point C in the diagram. All of the points on the circle, including X and Y, are the same distance from C, a distance called the *radius.*

This problem states that the diameter of circle C is 14. A *diameter* of a circle is a line segment with endpoints on the circle that passes through the center. Thus, $\overline{XY}$ is a diameter of the circle, because X and Y are points on the circle and the segment passes through center C. A diameter is exactly twice the length of a radius, so to calculate the radius of the circle, divide its diameter by two.

The diameter of C has length XY = 14. It is made up of two radii stuck together: XC + CY = XY. All radii have the same length, so XY is twice as long as XC and CY.

$$\begin{aligned}\text{radius} &= \text{diameter} \div 2\\ &= 14 \div 2\\ &= 7\end{aligned}$$

The radius of circle C is 7; this is typically written $r = 7$. Note that the two segments in the diagram that begin at C and end at points X and Y on the circle are radii of the circle, so $CX = CY = 7$.

Note: Problems 19.27–19.29 refer to the diagram in Problem 19.27, in which circle C has diameter 14 and points X and Y lie on the circle.

19.28 Calculate the area of circle C.

The area of a circle is equal to the square of its radius multiplied by the mathematical constant π, a nonrepeating, nonterminating decimal. Unless specifically directed otherwise by your instructor, do not estimate the value of π with a fraction or decimal—leave it in your final answer, as demonstrated in the following solution.

According to Problem 19.27, the radius of circle C is $r = 7$. Substitute it into the area formula for a circle.

$$\begin{aligned}A &= \pi r^2\\ &= \pi(7)^2\\ &= \pi(49)\\ &= 49\pi\end{aligned}$$

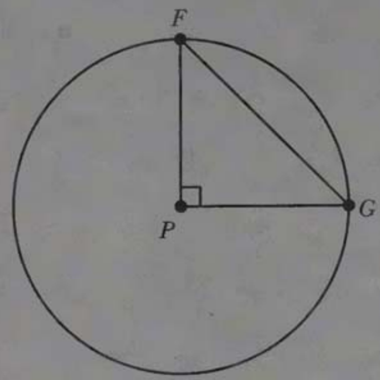
Note: Problems 19.27–19.29 refer to the diagram in Problem 19.27, in which circle C has diameter 14 and points X and Y lie on the circle.

19.29 Calculate the circumference of circle C.

The circumference of a circle is the equivalent of the perimeter of a triangle or quadrilateral—it is the total distance around the shape. It is used in lieu of the term "perimeter" to acknowledge that a perimeter is defined as the sum of the lengths of the sides of a figure, but a circle does not have sides.

The circumference of a circle is defined as the product of π and the diameter, so the circumference of circle C is 14π. Because the diameter of a circle is equal to twice the radius, it is common to express the formula as the product of π and $2r$: circumference $= 2\pi r$.

Note: Problems 19.30–19.32 refer to circle P in the diagram below. Points F and G lie on circle P and FG = 10.



19.30 Calculate the radius of circle P.

Segments $\overline{PF}$ and $\overline{PG}$ are radii of circle P, because each has one endpoint on the circle and the center of the circle as its other endpoint. Thus, both segments have length r. Notice that triangle FPG is a right triangle because the radii in the diagram are perpendicular. Apply the Pythagorean Theorem to calculate the radius of circle P.

$$\begin{aligned}(FP)^2+(PG)^2&=(FG)^2\\(r)^2+(r)^2&=(10)^2\\r^2+r^2&=100\\2r^2&=100\\\frac{2r^2}{2}&=\frac{100}{2}\\r^2&=50\\\sqrt{r^2}&=\sqrt{50}\end{aligned}$$

Simplify the radical expression, noting that 25 is a factor of 50 and is a perfect square: $25=5^2$.

$$\begin{aligned}r&=\sqrt{25\cdot 2}\\&=\sqrt{25}\cdot\sqrt{2}\\&=5\sqrt{2}\end{aligned}$$

Note: Problems 19.30–19.32 refer to the diagram in Problem 19.30, in which F and G lie on circle P and FG = 10.

19.31 Calculate the circumference of circle P.

According to Problem 19.30, the radius of circle P is $5\sqrt{2}$. Apply the circumference formula introduced in Problem 19.29.

$$\begin{aligned} C &= 2\pi r \\ &= 2\pi\left(5\sqrt{2}\right) \\ &= (2\cdot 5)\cdot\pi\cdot\sqrt{2} \\ &= 10\pi\sqrt{2} \end{aligned}$$

Note: Problems 19.30–19.32 refer to the diagram in Problem 19.30, in which F and G lie on circle P and FG = 10.

19.32 Calculate the area of circle P.

The area A of the circle is equal to the product of π and the square of the radius.

$$\begin{aligned} A &= \pi r^2 \\ &= \pi\left(5\sqrt{2}\right)^2 \end{aligned}$$

Square each of the factors within parentheses and simplify the expression.

$$\begin{aligned} &= \pi(5)^2\left(\sqrt{2}\right)^2 \\ &= \pi(25)\left(\sqrt{4}\right) \\ &= 25\pi(2) \\ &= 50\pi \end{aligned}$$

19.33 Calculate the radius of a circle with circumference $\frac{8}{3}\pi$.

Substitute $C = \frac{8}{3}\pi$ into the circumference formula and solve for r.

$$\begin{aligned} C &= 2\pi r \\ \frac{8}{3}\pi &= 2\pi r \end{aligned}$$

If you divide both sides of the equation by π, or multiply both sides by $\frac{1}{\pi}$, you eliminate π from the equation.

$$\begin{aligned} \frac{8}{3}\cdot\frac{\cancel{\pi}}{\cancel{\pi}} &= \frac{2\cancel{\pi}r}{\cancel{\pi}} \\ \frac{8}{3} &= \frac{2r}{1} \end{aligned}$$

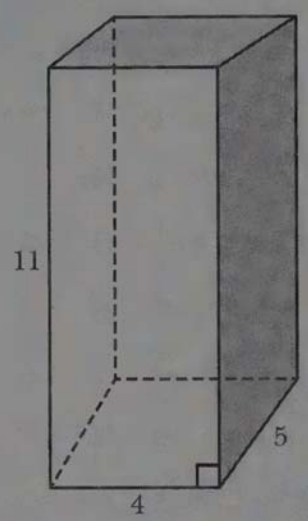
Cross-multiply to solve the proportion.

$$\begin{aligned} 8(1) &= 3(2r) \\ 8 &= 6r \\ \frac{8}{6} &= \frac{6r}{6} \\ \frac{8 \div 2}{6 \div 2} &= r \\ \frac{4}{3} &= r \end{aligned}$$

Solids

Volume of prisms, pyramids, cylinders, cones, and spheres

19.34 Calculate the volume of the rectangular prism below.



A prism is a three-dimensional solid, essentially a two-dimensional figure called the *base* that extends a distance called the *height*. In this problem, you are given a prism with a rectangular base; the base has length 5 and width 4. That rectangular base is then extended upwards to a height of 11, creating a solid rectangular box with rectangular sides that are perpendicular to the base.

Think of the solid as a whole bunch of rectangles with length 5 and width 4, stacked up to a height of 11.

If you slice the solid horizontally at any location, the cross-section will be a rectangle with the same dimensions as the base. To calculate the volume, multiply the area of the base (A_B) by the height (h).

$$V = A_B \cdot h$$

The base is a rectangle; multiply its length and width to calculate its area.

$$\begin{aligned} A_B &= lw \\ &= 5 \cdot 4 \\ &= 20 \end{aligned}$$

Substitute $A_B = 20$ and $h = 11$ into the volume formula to calculate the volume of the prism.

$$\begin{aligned} V &= A_B \cdot h \\ &= 20 \cdot 11 \\ &= 220 \end{aligned}$$

Textbooks often present the rectangular prism volume formula as $V = l \cdot w \cdot h$. The area of the base A_B is replaced with the area formula for a rectangle, $l \cdot w$. This is correct, but the more generic formula $V = A_B \cdot h$ is more useful when the base of the prism is not rectangular.

Like in Problem 19.36.

19.35 A rectangular prism with height 2 and volume 14 has a base with length 4. Calculate the width of the prism's base.

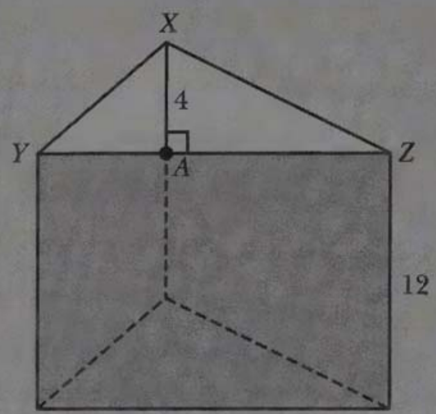
According to Problem 19.34, the formula $V = lwh$ computes the volume of a rectangular prism. You are given $V = 14$, $l = 4$, and $h = 2$. Substitute those values into the formula and solve for w.

$$\begin{aligned} V &= lwh \\ 14 &= (4)w(2) \\ 14 &= (4w)(2) \\ 14 &= 8w \end{aligned}$$

Divide both sides of the equation by 8 and reduce the fraction.

$$\begin{aligned} \frac{14}{8} &= \frac{8w}{8} \\ \frac{14}{8} &= w \\ \frac{14 \div 2}{8 \div 2} &= w \\ \frac{7}{4} &= w \end{aligned}$$

19.36 Calculate the volume of the prism below, given $YZ = 9$.



This prism has a triangular base, so you cannot apply the volume formula $V = lwh$. Rather, you must apply the more generic prism volume formula $V = A_B \cdot h$, in which A_B is the area of the prism's base and h is its height. The triangular base of this prism has base $YZ = 9$ and height $XA = 4$. Apply the triangle area formula.

A prism has two bases—one at the top and one at the bottom. You can use either one in the volume formula.

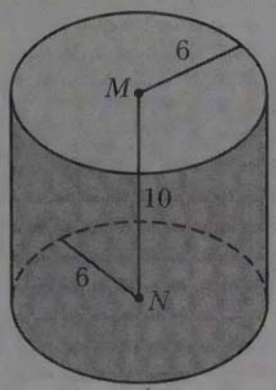
Don't get confused. The base of the prism is a triangle, and to calculate the area of a triangle, you take half of the triangle's base times the triangle's height.

$$\begin{aligned} A_B &= \frac{1}{2}bh \\ &= \frac{1}{2}(9)(4) \\ &= \left(\frac{9}{2}\right)\left(\frac{4}{1}\right) \\ &= \left(\frac{9}{1\cancel{2}}\right)\left(\frac{2\cancel{4}}{1}\right) \\ &= \left(\frac{9}{1}\right)\left(\frac{2}{1}\right) \\ &= 18 \end{aligned}$$

Substitute $A_B = 18$ and $h = 12$ into the prism volume formula.

$$\begin{aligned} V &= A_B \cdot h \\ &= 18 \cdot 12 \\ &= 216 \end{aligned}$$

19.37 Calculate the volume of the cylinder illustrated below.



A cylinder is very similar to a prism—both have a base that is extended to a fixed height, and a cross-section parallel to the base is exactly the same size as the base. For example, this cylinder has a circular base of radius 6, and if you were to slice the cylinder parallel to the base, the cross-section would also be a circle of radius 6. Like a prism, you calculate the volume V of a cylinder by multiplying the area of the base A_B by the height h.

$$V = A_B \cdot h$$

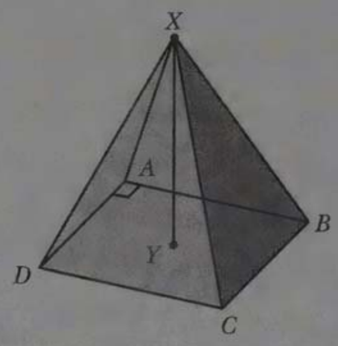
Apply the circle area formula to calculate the area of the base of the cylinder.

$$\begin{aligned} A_B &= \pi r^2 \\ &= \pi(6)^2 \\ &= \pi(36) \\ &= 36\pi \end{aligned}$$

Substitute A_B into the cylinder volume formula.

$$\begin{aligned} V &= A_B \cdot h \\ &= (36\pi)(10) \\ &= 360\pi \end{aligned}$$

19.38 Calculate the volume of the pyramid below, which has height 6 and square base *ABCD* with side length 3.



A *pyramid*, or *tetrahedron*, is a solid comprised of faces that extend from the sides of its base and meet at a single point a fixed distance away from the base called the *height*. In this problem, the pyramid rests on square base *ABCD*, and each side of the base has length 3. The faces of the pyramid (triangles *XDC*, *XCB*, *XBA*, and *XAD*) extend from the sides of the square to meet at point *X*; the pyramid has a height of $XY = 6$.

It's the prism volume formula multiplied by one-third.

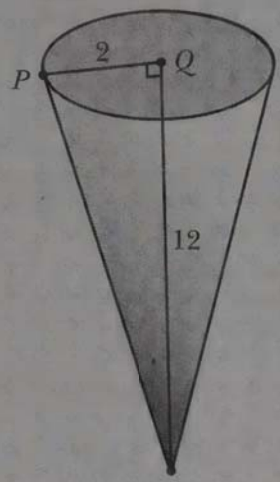
The formula for the volume of a pyramid is $V = \frac{1}{3}A_B h$, such that A_B is the area of the base and h is the height of the pyramid. Begin by calculating the area of the square base, given the length of a side is $s = 3$.

$$\begin{aligned} A_B &= s^2 \\ &= (3)^2 \\ &= 9 \end{aligned}$$

Substitute $A_B = 9$ and $h = 6$ into the pyramid volume formula to calculate *V*.

$$\begin{aligned} V &= \frac{1}{3}A_B h \\ &= \frac{1}{3}(9)(6) \\ &= \left(\frac{9}{3}\right)(6) \\ &= (3)(6) \\ &= 18 \end{aligned}$$

19.39 Calculate the volume of the cone illustrated below, which has a circular base with center Q and height 12.



You apply the same formula to calculate the volume of a cone and the volume of a pyramid: $V = \frac{1}{3}A_B h$. The base of this cone is a circle with radius $PQ = 2$. Calculate A_B, the area of the base.

$$\begin{aligned} A_B &= \pi r^2 \\ &= \pi(2)^2 \\ &= \pi(4) \\ &= 4\pi \end{aligned}$$

Substitute $A_B = 4\pi$ and $h = 12$ (the height of the solid) into the cone volume formula.

$$\begin{aligned} V &= \frac{1}{3}A_B h \\ &= \frac{1}{3}(4\pi)(12) \\ &= \left(\frac{4}{3}\pi\right)\left(\frac{12}{1}\right) \end{aligned}$$

Reduce the product before you multiply, noting that denominator value 3 and numerator value 12 have greatest common factor 3.

$$\begin{aligned} &= \left(\frac{4}{{}_1\cancel{3}}\pi\right)\left(\frac{{}^4\cancel{12}}{1}\right) \\ &= \left(\frac{4}{1}\pi\right)\left(\frac{4}{1}\right) \\ &= 16\pi \end{aligned}$$

19.40 Calculate the volume of a sphere with radius 3.

To calculate the volume of a sphere with radius r, apply the formula $V = \frac{4}{3}\pi r^3$.

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi (3)^3 \\ &= \frac{4}{3}\pi (27) \\ &= \frac{4}{3}\pi \left(\frac{27}{1}\right) \end{aligned}$$

Reduce the product to lowest terms and then multiply.

$$\begin{aligned} &= \frac{4}{1\,\cancel{3}}\pi \left(\frac{9\,\cancel{27}}{1}\right) \\ &= \frac{4}{1}\pi \left(\frac{9}{1}\right) \\ &= 36\pi \end{aligned}$$

Chapter 20
WEIGHTS AND MEASURES

With a cameo by the metric system

The world's most ubiquitous system of measurement is the metric system. The most notable exception to the worldwide adoption of the metric system as a standard is the United States, which prefers the English system of measurement (although scientific measurements are typically reported in metric units, even in the U.S.). The two systems are vastly different. While the metric system assigns one base unit for each kind of measure (for example, volume is expressed in liters and distance in meters) and modifies them with prefixes (including kilo-, centi-, and milli-), the English system uses a vast array of units (including the foot, mile, pint, quart, gallon, ounce, and pound, to name a few), and no defining prefixes.

In this chapter, you will first convert between units in one system (grams to kilograms, ounces to pounds) and then convert between the two systems of measurement. The latter task is not complex, but it is arithmetically intensive. Using a calculator to complete the multiplication portions of these problems is not only permitted, it is encouraged. Otherwise, these exercises are merely tests of multiple-digit multiplication.

Do you know how many meters are in a mile? How about the numbers of grams in a pound? Are tons and metric tons the same thing? Each of these questions deals with two different systems of measurement, and to convert from one system to the other, you need to know a secret conversion number to multiply (or divide) by. That's all there is to it. By the way, all of the important conversion formulas are listed at the end of the book in Appendix D.

The book doesn't use abbreviations for the units (lb instead of pounds, mL instead of milliliters) until Problems 20.29–20.42. They appear in the big chart before Problem 20.29.

Converting Units of Time

Practice converting between units

20.1 Calculate the number of seconds in an hour.

One hour contains 60 minutes, and each of those 60 minutes contains 60 seconds.

$$\begin{aligned}\text{seconds} &= (\text{hours})(\text{minutes per hour})(\text{seconds per minute})\\ &= (1)(60)(60)\\ &= 3{,}600\end{aligned}$$

There are 3,600 seconds in one hour.

20.2 Calculate the number of hours in one week.

One week consists of seven days. Each of those days contains 24 hours.

$$\begin{aligned}\text{hours} &= (\text{weeks})(\text{days per week})(\text{hours per day})\\ &= (1)(7)(24)\\ &= 168\end{aligned}$$

Thus, there are 168 hours in one week.

20.3 Calculate the number of full, seven-day weeks in one (non-leap) year. Round your answer to the nearest hundredth.

In a leap year, February has 29 days instead of 28, so there are 366 days instead of 365.

There are 365 days in a non-leap year and there are seven days in a week. To calculate the total number of weeks in a year, divide 365 by 7.

$$365 \div 7 = 52.\overline{142857}$$

The horizontal bar above the digits 142857 indicates that those digits repeat infinitely. The problem directs you to round to the hundredths place, two digits right of the decimal point: $365 \div 7 \approx 52.14$. To review rounding, see Problems 10.8–10.14.

20.4 Calculate the number of seconds in one day.

A day contains 24 hours. Each of those 24 hours contains 60 minutes, and each of those minutes contains 60 seconds. Multiply each of the units to calculate the answer.

$$\begin{aligned}1 \text{ day} &= (\text{hours per day})(\text{minutes per hour})(\text{seconds per minute})\\ &= (24)(60)(60)\\ &= (1{,}440)(60)\\ &= 86{,}400\end{aligned}$$

There are 86,400 seconds in one day.

20.5 Calculate the number of minutes in one (non-leap) year.

You already know the answer if you're familiar with the song "Seasons of Love" from the musical Rent.

One non-leap year contains 365 days, each of which contains 24 hours. Each hour contains 60 minutes. Multiply these values to solve the problem.

$$\begin{aligned} 1 \text{ year} &= (\text{days per year})(\text{hours per day})(\text{minutes per hour}) \\ &= (365)(24)(60) \\ &= (8{,}760)(60) \\ &= 525{,}600 \end{aligned}$$

20.6 Calculate the number of days in one week and the number of weeks in one day. Round your answer to the nearest hundredth.

A week contains seven days. Because a week represents a longer period of time than a single day, expressing the number of weeks in a single day will produce a decimal value less than one, a value equal to the reciprocal of the relationship. For example, there are 7 days in a week, so there are $\frac{1}{7}$ weeks in a day.

$$1 \div 7 = 0.\overline{142857}$$

Round your answer to the nearest hundredth.

$$1 \div 7 \approx 0.14$$

There are approximately 0.14 weeks in one day.

English System

Inches, feet, pints, gallons, ounces, and pounds

20.7 Convert the distance into feet: 7 yards.

One yard is exactly three feet in length. This problem presents a distance of 7 yards, each of which is 3 feet long. Multiply 7 by 3 to express the distance in terms of feet.

$$\begin{aligned} \text{feet} &= (\text{yards})(\text{feet per yard}) \\ &= (7)(3) \\ &= 21 \end{aligned}$$

Therefore, 7 yards and 21 feet represent the same distance.

20.8 Convert the distance into inches: 5 feet.

And a distance expressed in feet is one-twelfth the size of the same distance expressed in inches.

One foot contains 12 inches. Thus, a distance of 5 feet is equal to $5 \cdot 12 = 60$ inches. Because an inch is exactly one-twelfth the size of a foot, a distance expressed in inches is 12 times as large as the same distance expressed in feet.

20.9 Convert the distance into inches: 3.5 miles.

One mile measures 5,280 feet, and each of those feet contains 12 inches. Multiply the distance (expressed in miles) by the number of feet per mile and the number of inches per foot to express the distance in inches.

$$\begin{aligned} 3.5 \text{ miles} &= (\text{distance in miles})(\text{feet per mile})(\text{inches per foot}) \\ &= (3.5)(5{,}280)(12) \\ &= (18{,}480)(12) \\ &= 221{,}760 \end{aligned}$$

A distance of 3.5 miles is equivalent to a distance of 221,760 inches.

20.10 Calculate the number of fluid ounces in 4 cups.

One cup contains 8 fluid ounces.

$$\begin{aligned} \text{fluid ounces} &= (\text{cups})(\text{fluid ounces per cups}) \\ &= (4)(8) \\ &= 32 \end{aligned}$$

You conclude that 4 cups is equal to 32 fluid ounces.

So a half-pint is equal to one cup.

20.11 Calculate the number of cups in 3 pints.

One pint represents the same volume as two cups. Therefore, a volume of 3 pints is equal to $3(2) = 6$ cups.

20.12 Convert the volume into gallons: 7 quarts.

One gallon contains four quarts. However, you do not multiply 7 quarts by 4 to calculate the number of gallons—the result would be 28. Gallons are larger than quarts, so concluding that a volume of 7 quarts is equal to a volume of 28 gallons is illogical.

As Problem 20.6 explains, when you convert from a smaller unit to a larger unit—like in this problem, when you convert from quarts to gallons—you *divide* rather than multiply. If there are 4 quarts in a gallon, then each quart is $1 \div 4 = \frac{1}{4}$ of a gallon.

$$\begin{aligned}(7\text{ quarts})\div(\text{quarts per gallon}) &= \text{gallons}\\ 7\div 4 &= \text{gallons}\\ \frac{7}{4} &= \text{gallons}\end{aligned}$$

Therefore, 7 quarts and $\frac{7}{4}$ gallons represent the same volume. It is also correct to express your answer as a decimal: 1.75 gallons.

20.13 Calculate the number of fluid ounces in 10 gallons.

One gallon contains 4 quarts, each quart contains 2 pints, each pint contains 2 cups, and each cup contains 8 fluid ounces. Multiply those values to calculate the number of fluid ounces in *one* gallon.

$$\begin{aligned}&(\text{quarts per gallon})(\text{pints per quart})(\text{cups per pint})(\text{fl. ounces per cup})\\ &=(4)(2)(2)(8)\\ &=128\end{aligned}$$

Therefore, one gallon contains 128 fluid ounces. Multiply by 10 to calculate the number of fluid ounces in 10 gallons.

$$\begin{aligned}10\text{ gallons} &= 10(\text{fluid ounces per gallon})\\ &=10(128)\\ &=1{,}280\text{ fluid ounces}\end{aligned}$$

20.14 Which represents the greater volume: 47.5 cups or 3 gallons?

According to Problem 20.13, one gallon contains 128 fluid ounces. Therefore, 3 gallons contain $3(128) = 384$ fluid ounces. Recall that one cup contains 8 fluid ounces. Thus, 47.5 cups contain $47.5(8) = 380$ fluid ounces. Three gallons (384 fluid ounces) represents a larger volume than 47.5 cups (380 fluid ounces).

Ounces (and pounds) measure weight. Fluid ounces (and cups, pints, quarts, and gallons) measure volume.

20.15 Express a weight of 120 ounces in pounds.

One pound contains 16 ounces. In this problem, you convert from a smaller unit (ounces) to a larger unit (pounds), so you should *divide by* 16 rather than *multiply by* 16—each of the 120 ounces represents $\frac{1}{16}$ of a pound.

$$\begin{aligned}\text{pounds} &= \text{ounces}\div(\text{ounces per pound})\\ &=120\div 16\\ &=\frac{120}{16}\\ &=\frac{120\div 8}{16\div 8}\\ &=\frac{15}{2}\end{aligned}$$

You conclude that 120 ounces is equal to $\frac{15}{2}$, or 7.5, pounds.

20.16 Convert the weight into pounds: 8 tons.

One ton is equal to 2,000 pounds. Thus, 8 tons is equal to $8(2{,}000) = 16{,}000$ pounds.

20.17 Convert the weight into tons: 3,500 pounds.

According to Problem 20.16, one ton contains 2,000 pounds. Therefore, one pound is equal to $\frac{1}{2{,}000}$ of a ton. Divide 3,500 by 2,000 to convert the weight into tons.

$$\begin{aligned} \text{tons} &= \text{pounds} \div 2{,}000 \\ &= 3{,}500 \div 2{,}000 \\ &= \frac{3{,}500}{2{,}000} \\ &= \frac{3{,}500 \div 500}{2{,}000 \div 500} \\ &= \frac{7}{4} \end{aligned}$$

You conclude that 3,500 pounds is equal to $\frac{7}{4}$, or 1.75, tons.

Metric System

Meters, liters, grams, and a whole lot of prefixes

20.18 Identify the metric prefixes that correspond to the following values: (a) 1,000; (b) 100; (c) 10; (d) $\frac{1}{10}$; (e) $\frac{1}{100}$; and (f) $\frac{1}{1{,}000}$.

The metric system uses prefixes to describe magnitudes of measure. For instance, a kilogram is equal to 1,000 grams because the prefix "kilo-" represents 1,000. It, and five of the other most common prefixes, are listed in the chart below.

Prefix	*Magnitude*
kilo	1,000
hecta	100
deca	10
deci	1/10
centi	1/100
milli	1/1,000

Some books spell this one "deka"; both are pronounced "DECK-uh."

20.19 Convert the distance into meters: 4.2 kilometers.

According to Problem 20.18, the prefix "kilo-" corresponds with 1,000. In the context of this problem, a kilometer is equal to 1,000 meters. Multiply 4.2 kilometers by 1,000 to convert the distance into meters.

$$\begin{aligned} \text{meters} &= (\text{kilometers})(\text{meters per kilometer}) \\ &= (4.2)(1{,}000) \\ &= 4{,}200 \end{aligned}$$

Therefore, 4.2 kilometers = 4,200 meters.

20.20 Convert the distance into meters: 4.2 millimeters.

The prefix "milli-" corresponds with one one-thousandth $\left(\frac{1}{1{,}000}\right)$. One millimeter is equal to one one-thousandth of a meter; in other words, one meter is equal to one thousand millimeters. To convert from millimeters into meters, divide by 1,000.

$$\begin{aligned} \text{meters} &= (\text{millimeters}) \div (\text{millimeters per meter}) \\ &= \frac{4.2}{1{,}000} \\ &= 0.0042 \end{aligned}$$

Therefore, 4.2 millimeters is equal to 0.0042 meters.

Check to see if your answer makes sense. One meter equals 1,000 millimeters, so 4.2 millimeters is a very small distance, much smaller than one meter. An answer like 4.2(1,000) = 4,200 wouldn't make sense.

20.21 Convert the distance into decameters: 700 centimeters.

One meter contains 100 centimeters; the prefix "centi-" corresponds with the value one one-hundredth $\left(\frac{1}{100}\right)$. To express 700 centimeters in terms of meters, divide by 100.

$$\begin{aligned} \text{meters} &= (\text{centimeters}) \div (\text{centimeters per meter}) \\ &= 700 \div 100 \\ &= 7 \end{aligned}$$

The problem directs you to express your answer in decameters. According to Problem 20.18, one decameter is equal to 10 meters. You are converting from a smaller to a larger unit, so divide by 10. (If you were converting from decameters to meters—from a larger to a smaller unit, you would multiply by 10.)

$$\begin{aligned} \text{decameters} &= (\text{meters}) \div (\text{meters per decameter}) \\ &= 7 \div 10 \\ &= \frac{7}{10} \end{aligned}$$

You conclude that a distance of 700 centimeters is equal to $\frac{7}{10}$ or 0.7 decameters.

20.22 Convert the volume into milliliters: 8.21 liters.

According to Problem 20.20, one meter contains 1,000 millimeters. Although those units measure distance, a similar relationship exists for other units of measure and the "milli-" prefix. For example, one liter contains 1,000 milliliters. Therefore, multiplying 8.21 liters by 1,000 converts the volume into milliliters.

$$\begin{aligned}\text{milliliters} &= (\text{liters})(\text{milliliters per liter}) \\ &= (8.21)(1{,}000) \\ &= 8{,}210\end{aligned}$$

You conclude that 8.21 liters is equal to 8,210 milliliters.

20.23 Convert the volume into milliliters: 950 microliters.

There are different methods you can apply to convert between different metric units, but because metric prefixes are all defined in terms of the base unit (in this case, liters), it is often helpful to convert to that base unit as an intermediate step.

In other words, convert from microliters to liters and then convert liters to milliliters.

One liter contains 1,000,000 microliters. To convert from the smaller unit microliters to the larger unit liters, divide by 1,000,000.

$$\begin{aligned}\text{liters} &= (\text{microliters}) \div (\text{microliters per liter}) \\ &= \frac{950}{1{,}000{,}000} \\ &= \frac{950 \div 50}{1{,}000{,}000 \div 50} \\ &= \frac{19}{20{,}000} \\ &= 0.00095\end{aligned}$$

Therefore, 950 microliters is equal to 0.00095 liters. Recall that one liter contains 1,000 milliliters. As Problem 20.22 explains, multiplying by 1,000 converts from the larger unit liters to the smaller unit milliliters.

$$\begin{aligned}\text{milliliters} &= (\text{liters})(\text{milliliters per liter}) \\ &= (0.00095)(1{,}000) \\ &= 0.95\end{aligned}$$

20.24 Convert the volume into kiloliters: 2,900 deciliters.

As demonstrated in Problem 20.23, it is often helpful to convert to the base metric unit of measure as an intermediate step. First convert deciliters to liters and then convert the result into kiloliters.

One liter contains 10 deciliters. To convert from the smaller unit deciliters to the larger unit liters, divide by 10.

$$\begin{aligned}\text{liters} &= (\text{deciliters}) \div (\text{deciliters per liter})\\ &= (2{,}900) \div 10\\ &= 290\end{aligned}$$

One kiloliter contains 1,000 liters. Divide by 1,000 to convert from the smaller unit liters to the larger unit kiloliters.

$$\begin{aligned}\text{kiloliters} &= (\text{liters}) \div (\text{liters per kiloliter})\\ &= \frac{290}{1{,}000}\\ &= \frac{290 \div 10}{1{,}000 \div 10}\\ &= \frac{29}{100}\\ &= 0.29\end{aligned}$$

You conclude that 2,900 deciliters is equal to 0.29 kiloliters.

20.25 Describe the difference between an object's weight and its mass.

The *mass* of an object measures the amount of material in the object. Mass is constant regardless of the object's location. For example, a bowling ball on Earth has the same mass as that bowling ball if it were lying on the surface of the moon.

The *weight* of an object describes the pull of gravity on that object. To build upon the previous metaphor, a bowling ball on Earth weighs more than it does on the moon, because the force of gravity is stronger on Earth.

In this chapter, you will assume that all measurements are taken under the same force of gravity, so weight and mass are roughly equivalent. Although they describe fundamentally different characteristics of an object, it is not inappropriate to convert between units of mass and units of weight. This is important because English units such as ounces, pounds, and tons measure the weight of an object, whereas metric units such as milligrams, grams, and kilograms measure an object's mass.

20.26 Convert the mass into grams: 225 kilograms.

One kilogram contains 1,000 grams. Multiply by 1,000 to convert from the larger unit kilograms to the smaller unit grams.

$$\begin{aligned}\text{grams} &= (\text{kilograms})(\text{grams per kilogram})\\ &= (225)(1{,}000)\\ &= 225{,}000\end{aligned}$$

You conclude that 225 kilograms is equal to 225,000 grams.

20.27 Convert the mass into picograms: 16 grams.

The prefix "pico-" corresponds with one one-trillionth $\left(10^{-12} = \dfrac{1}{1{,}000{,}000{,}000{,}000}\right)$. Therefore, each gram contains one trillion pictograms. To convert from grams to pictograms, multiply by 10^{12}.

$$\begin{aligned}\text{picograms} &= (\text{grams})(\text{picograms per metric ton})\\ &= 16(1{,}000{,}000{,}000{,}000)\\ &= 16{,}000{,}000{,}000{,}000\end{aligned}$$

You conclude that 16 grams is equal to 16 trillion picograms.

20.28 Convert the mass into metric tons: 500 kilograms.

The metric system refers to English system tons as "short tons" to avoid confusion with the metric ton.

According to Problem 20.16, one ton in the English system is equal to 2,000 pounds. In the metric system, one *ton* (or *tonne*) is equal to 1,000 kilograms. Divide by 1,000 to convert from the smaller unit kilograms to the larger unit metric tons.

$$\begin{aligned}\text{metric tons} &= (\text{kilograms}) \div (\text{kilograms per metric ton})\\ &= \frac{500}{1{,}000}\\ &= \frac{500 \div 500}{1{,}000 \div 500}\\ &= \frac{1}{2}\\ &= 0.5\end{aligned}$$

You conclude that 500 kilograms equals 0.5 metric tons. Because a kilogram is equal to 1,000 grams, a metric ton contains 1,000,000 grams. The prefix "mega-" corresponds with a value of one million, so a metric ton is also known as a *megagram*. Therefore, a final answer of 0.5 megagrams is also correct.

Converting Between Metric and English Units

Including Celsius and Fahrenheit

Note: The values in the table below identify equal measures in the English and metric systems. To convert from English to metric units, use the constants in the left column; use the constants in the right column to convert from metric to English units.

The following abbreviations are used in the English system: ounce (oz), pound (lb), ton (t), fluid ounce (fl oz), cup (c), pint (pt), quart (qt), gallon (gal), inch (in), foot (ft), yard (yd), mile (mi). The following abbreviations are used in the metric system: gram (g), milligram (mg), kilogram (kg), metric ton (MT)/megagram (Mg), liter (L), milliliter (mL), meter (m), centimeter (cm), millimeter (mm), kilometer (km).

	English → Metric	*Metric → English*
Weight / Mass	1 oz = 28.3495 g	1 g = 0.0353 oz
	1 lb = 0.4536 kg	1 kg = 2.2046 lb
	1 t = 0.9072 Mg	1 Mg = 1.1023 t
Volume	1 fl oz = 29.5735 mL	1 L = 1.0567 qt
	1 qt = 0.9464 L	1 L = 0.2642 gal
	1 gal = 3.7854 L	1 mL = 0.0338 fl oz
Distance	1 in = 2.54 cm	1 cm = 0.3937 in
	1 ft = 30.48 cm	1 cm = 0.0328 ft
	1 yd = 0.9144 m	1 m = 1.0936 yd
	1 mi = 1.6093 km	1 km = 0.6214 mi

20.29 Convert 7 oz into grams and round your answer to the hundredths place.

You'll round most of the answers in this chapter.

According to the preceding table, one ounce is approximately equal to 28.3495 g. To convert a weight expressed in ounces into a mass expressed in grams multiply by 28.3495.

$$\begin{aligned}\text{grams} &= (\text{ounces})(\text{g per oz})\\ &= (7)(28.3495)\\ &= 198.4465\end{aligned}$$

Note that the conversion value 28.3495 is not precise—rather, it is rounded to the ten-thousandths place. It is more precise and more accurate to say that 1 oz = 28.3495231 g, but it is rare that you would need such an accurate answer in non-scientific applications. Thus, it is appropriate to use the shorter conversion value 28.3495.

To acknowledge the impreciseness of the conversion calculations between different systems of measure, you are directed to round to the hundredths place. You conclude that 7 oz ≈ 198.45 g.

20.30 Convert 4.1 Mg into tons and round your answer to the hundredths place.

As explained in Problem 20.28, a metric ton is also called a megagram (Mg). To convert from megagrams to tons, multiply by 1.1023. (This value appears in the right column of the chart that precedes Problem 20.29.)

$$\begin{aligned} \text{tons} &= (\text{megagrams})(\text{t per Mg}) \\ &= (4.1)(1.1023) \\ &= 4.51943 \end{aligned}$$

You conclude that 4.1 Mg ≈ 4.52 t.

20.31 Convert 828 kg into ounces and round your answer to the nearest whole number.

According to the chart that precedes Problem 20.29, 1 kg ≈ 2.2046 lb. Each of the 828 kg is approximately equal to 2.2046 lb, and each pound contains 16 oz.

$$\begin{aligned} \text{ounces} &= (\text{kilograms})(\text{lb per kg})(\text{oz per lb}) \\ &= (828)(2.2046)(16) \\ &= (1825.4088)(16) \\ &= 29{,}206.5408 \end{aligned}$$

You conclude that 828 kg ≈ 29,207 oz.

The abbreviation for pounds is lb, short for "libre pondo," which is Latin for "pound of weight."

20.32 In Great Britain and Ireland, people commonly report their weight in terms of the unit *stone*, which is equal to 14 lb. Convert the weight 13 stone into kilograms and round your answer to the hundredths place.

Each stone contains 14 lb, and according to the conversion chart preceding Problem 20.29, 1 lb = 0.4536 kg.

$$\begin{aligned} \text{kilograms} &= (\text{stone})(\text{lb per stone})(\text{kg per lb}) \\ &= (13)(14)(0.4536) \\ &= 182(0.4536) \\ &= 82.5552 \end{aligned}$$

Thus, 13 stone ≈ 82.56 kg.

Don't make unit abbreviations plural. Two liters is written 2 L, not 2 Ls. The answer is 0.53 gal, not 0.53 gals. It's okay to put periods after the abbreviations (2 L. and 0.53 gal.), but most textbooks don't.

20.33 Calculate the volume, in gallons, of a two-liter soft drink bottle and round your answer to the hundredths place.

Multiply a volume measured in liters by 0.2642 to convert it to gallons.

$$\begin{aligned} \text{gallons} &= (\text{liters})(\text{gal per L}) \\ &= 2(0.2642) \\ &= 0.5284 \end{aligned}$$

Therefore, 2 L ≈ 0.53 gal.

20.34 Convert 5 mL into fluid ounces and round your answer to the thousandths place.

According to the chart preceding Problem 20.29, 1 mL = 0.0338 fl oz.

$$\begin{aligned}\text{fluid ounces} &= (\text{milliliters})(\text{fl oz per mL})\\ &= (5)(0.0338)\\ &= 0.169\end{aligned}$$

You conclude that 5 mL ≈ 0.169 fl oz.

20.35 Convert 11 c into liters and round your answer to the hundredths place.

The chart preceding Problem 20.29 states that 1 qt = 0.9464 L. To apply this conversion, you must first convert 11 cups into quarts. Recall that each cup represents half of a pint (because one pint contains two cups).

$$\begin{aligned}\text{pints} &= (\text{cups}) \div (\text{c per pt})\\ &= 11 \div 2\\ &= \frac{11}{2}\end{aligned}$$

Divide by two again to convert pints into quarts.

$$\begin{aligned}\text{quarts} &= (\text{pints}) \div (\text{pt per qt})\\ &= \frac{11}{2} \div 2\\ &= \frac{11}{2} \cdot \frac{1}{2}\\ &= \frac{11}{4}\\ &= 2.75\end{aligned}$$

To divide when fractions are involved, take the reciprocal of the number you're dividing by and change the division symbol to multiplication.

Thus, 11 c = 2.75 qt. Multiply by 0.9464 to convert to liters.

$$\begin{aligned}\text{liters} &= (\text{quarts})(\text{qt per L})\\ &= (2.75)(0.9464)\\ &= 2.6026\end{aligned}$$

You conclude that 11 c ≈ 2.60 L.

20.36 Convert 3 pt into milliliters and round your answer to the hundredths place.

One pint contains two cups, each cup contains eight fluid ounces, and according to the chart preceding Problem 20.29, 1 fl oz = 29.5735 mL. Multiply as directed below to convert from pints to milliliters.

$$\begin{aligned}\text{milliliters} &= (\text{pints})(\text{c per pt})(\text{fl oz per cup})(\text{mL per fl oz})\\ &= (3)(2)(8)(29.5735)\\ &= 48(29.5735)\\ &= 1{,}419.528\end{aligned}$$

You conclude that 3 pt ≈ 1,419.53 mL.

20.37 Convert 15 cm into inches and round your answer to the hundredths place.

Refer to the chart preceding Problem 20.29 to determine that 1 cm = 0.3937 in.

$$\begin{aligned}\text{inches} &= (\text{centimeters})(\text{in per cm})\\ &= (15)(0.3937)\\ &= 5.9055\end{aligned}$$

You conclude that 15 cm ≈ 5.91 in.

20.38 Calculate the difference (in meters) of the lengths of two athletic fields, if one of the fields measures 100 y and the other measures 100 m. Round your answer to the nearest hundredth.

Multiply a length measured in yards by 0.9144 to convert it to meters.

$$\begin{aligned}\text{meters} &= (\text{yards})(\text{m per yd})\\ &= (100)(0.9144)\\ &= 91.44\end{aligned}$$

Now that both lengths are expressed in meters, you can calculate the difference.

$$\begin{aligned}100\text{ m} - 100\text{ y} &= 100\text{ m} - 91.44\text{ m}\\ &= 8.56\text{ m}\end{aligned}$$

20.39 Convert a height of 5 feet 9 inches to meters and round your answer to the nearest hundredth.

The chart doesn't contain a formula to convert from feet to meters, but you can convert from feet to centimeters and then convert from centimeters to meters.

One foot contains 12 inches, so 9 inches represents $\frac{9}{12} = 0.75$ ft. Therefore, a height of 5 feet 9 inches is equal to 5.75 feet. According to the chart preceding Problem 20.29, 1 ft = 30.48 cm.

$$\begin{aligned}\text{centimeters} &= (\text{feet})(\text{cm per ft})\\ &= (5.75)(30.48)\\ &= 175.26\end{aligned}$$

One meter contains 100 cm. To convert from the smaller unit centimeters to the larger unit meters, divide by 100.

$$\begin{aligned}\text{meters} &= (\text{centimeters}) \div (\text{cm per m})\\ &= 175.26 \div 100\\ &= 1.7526\end{aligned}$$

You conclude that a length of 5 ft 9 in is approximately equal to 1.75 m.

20.40 Light travels at a speed of 299,792,458 meters per second. Express the speed in miles per second, and round your answer to the nearest hundred.

This time you round to the nearest HUNDRED, not the nearest HUNDREDTHS. The bigger the numbers you're converting, the less accurate the conversion will be. To get more accurate answers, you'd have to use a decimal longer than 0.6214.

Convert 299,792,458 m/sec to km/sec, noting that 1 km = 1,000 m.

$$299{,}792{,}458 \text{ km/sec} \div 1{,}000 = 299{,}792.458 \text{ km/sec}$$

To convert from kilometers to miles, multiply by 0.6214.

$$\begin{aligned}\text{miles} &= (\text{kilometers})(\text{mi per km})\\ &= (299{,}792.458)(0.6214)\\ &= 186291.0334012\end{aligned}$$

You are instructed to round your answer to the nearest hundred, so you conclude that the speed of light is approximately 186,300 mi/sec.

20.41 Water boils at a temperature of 100 degrees Celsius (100°C). Calculate the boiling point of water in degrees Fahrenheit (°F).

Apply the formula below, in which c represents degrees Celsius and f represents degrees Fahrenheit, to convert from the metric unit for temperature to the English unit for temperature.

$$f = \frac{9}{5}c + 32$$

In this problem, $c = 100$. Calculate the corresponding value of f.

$$\begin{aligned}f &= \frac{9}{5}(100) + 32\\ &= \frac{9}{5}\left(\frac{100}{1}\right) + 32\\ &= \frac{9}{1\,\cancel{5}}\left(\frac{20\,\cancel{100}}{1}\right) + 32\\ &= \frac{9}{1}\left(\frac{20}{1}\right) + 32\\ &= 180 + 32\\ &= 212\end{aligned}$$

You conclude that 100°C is equal to 212°F.

20.42 Convert the temperature into degrees Celsius: 95°F.

Problem 20.41 presents a formula to convert from Fahrenheit to Celsius. To convert from Celsius to Fahrenheit, solve that formula for f. You will need to subtract 32 from both sides of the equation and then multiply everything by the coefficient of c.

$$\begin{aligned} f &= \frac{9}{5}c + 32 \\ f - 32 &= \frac{9}{5}c + (32 - 32) \\ f - 32 &= \frac{9}{5}c \\ \frac{5}{9}(f - 32) &= \frac{5}{9}\left(\frac{9}{5}\right)c \\ \frac{5}{9}(f - 32) &= c \end{aligned}$$

Substitute $f = 95$ into the formula to calculate the corresponding value of c.

$$\begin{aligned} \frac{5}{9}(95 - 32) &= c \\ \frac{5}{9}(63) &= c \\ \frac{5}{9}\left(\frac{63}{1}\right) &= c \\ \frac{5}{{}_1\cancel{9}}\left(\frac{{}^7\cancel{63}}{1}\right) &= c \\ \frac{5}{1}\left(\frac{7}{1}\right) &= c \\ 35 &= c \end{aligned}$$

You conclude that 95°F = 35°C.

Chapter 21
BASIC STATISTICS

Visualizing and analyzing data

You are presented with collections of data every day, and you routinely make decisions based upon the way you interpret them. Whether you are packing vacation clothing based on weather predictions or monitoring your progress in an academic course based upon your quiz and test scores, data offers you numeric snapshots in time. The appropriate use of statistics helps you characterize and visualize the data without having to study each of the values individually, allowing you to view the proverbial forest without having to inspect each tree.

Full statistics courses dedicate much of their time to hypothesis testing, including research studies that attempt to validate claims based upon experimental data, but that is far too complex to investigate within the scope of this book. Instead, this chapter introduces basic statistical concepts and explores different ways to present collections of data visually.

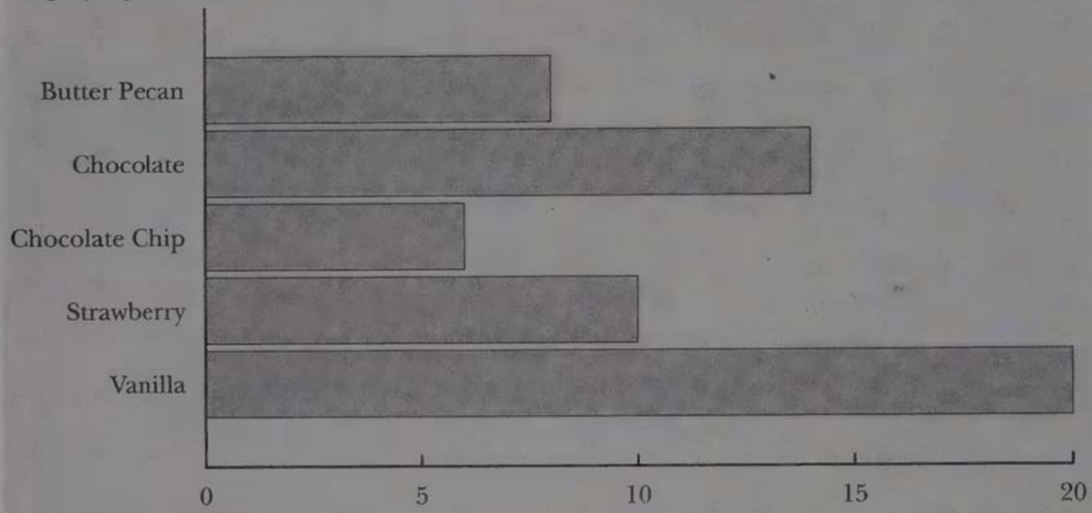
Statistical tools allow you to look at a giant pile of numbers and draw conclusions about those numbers. For example, academic classes constantly collect data when you take tests and quizzes. If a teacher gives a test, it is important to ask, "What do these scores mean?"

To analyze the scores, the teacher might construct a bar chart that compares the number of As, Bs, Cs, Ds, and Fs that were earned. He or she might also calculate the central tendency of the students: How did the "average" student perform? It may also be helpful to divide the scores into four chunks, each containing 25% of the data to see how similarly or differently students performed. You'll learn how to do all of these things (and more) in this chapter.

Interpreting Charts and Graphs

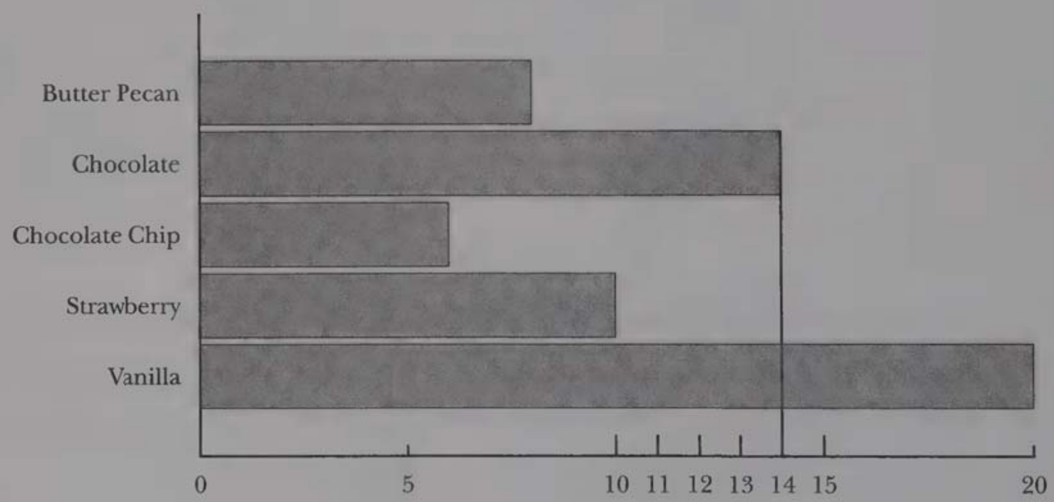
Bar, pie, and scatter charts

Note: Problems 21.1–21.2 refer to the diagram below, a horizontal column bar chart displaying the total sales of ice cream cones at a diner during lunch on a hot afternoon.



21.1 How many chocolate ice cream cones were sold?

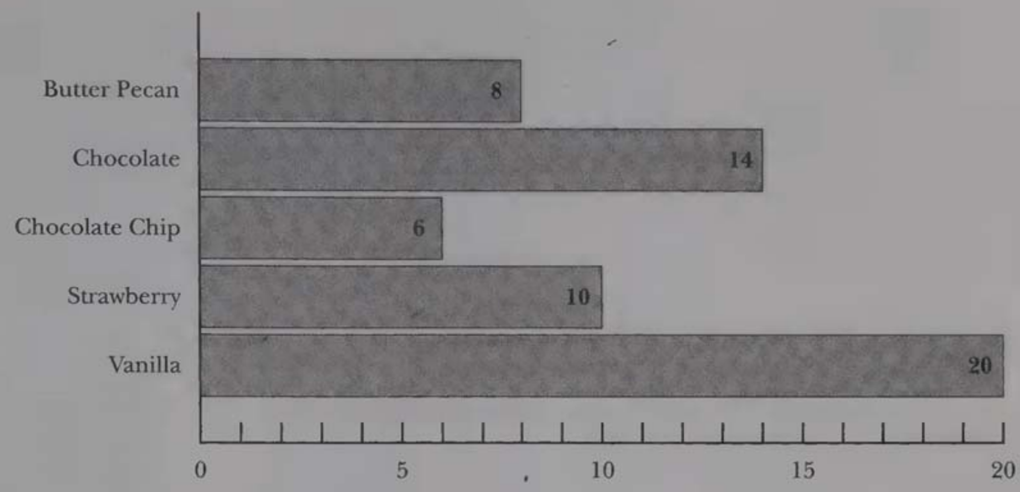
The second horizontal bar from the top represents the number of chocolate ice cream cones. It stretches a total distance of 14 units from the vertical axis, as illustrated below. Therefore, a total of 14 chocolate cones were sold.



Note: Problems 21.1–21.2 refer to the diagram in Problem 21.1, a bar chart displaying the total sales of ice cream cones at a diner during lunch on a hot afternoon.

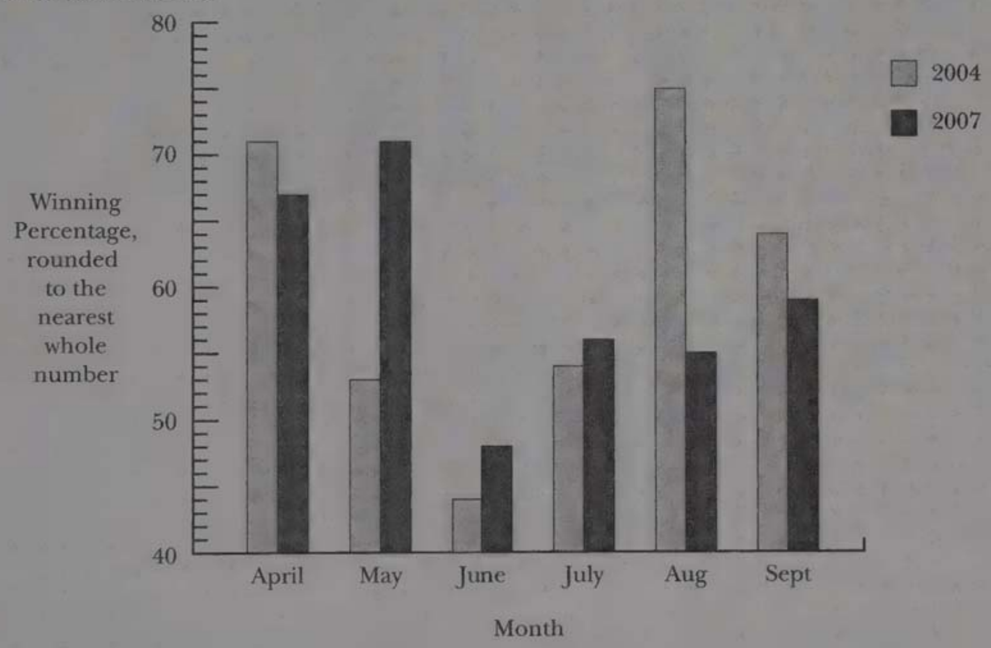
21.2 List the flavors in order of total sales, from least to greatest.

Calculate the length of each horizontal bar to determine the total sales for each flavor, as illustrated in the following graph.



The total sales per flavor, from least to greatest, are: chocolate chip (6), butter pecan (8), strawberry (10), chocolate (14), and vanilla (20).

Note: Problems 21.3–21.4 refer to the diagram below, a grouped vertical bar chart that compares the winning percentages of the Boston Red Sox each month during two years they won the World Series.



Notice that the vertical axis (winning percentage) starts at 40%, not 0%. The team's winning percentage did not fall below 40% for any month during the 2004 or 2007 seasons.

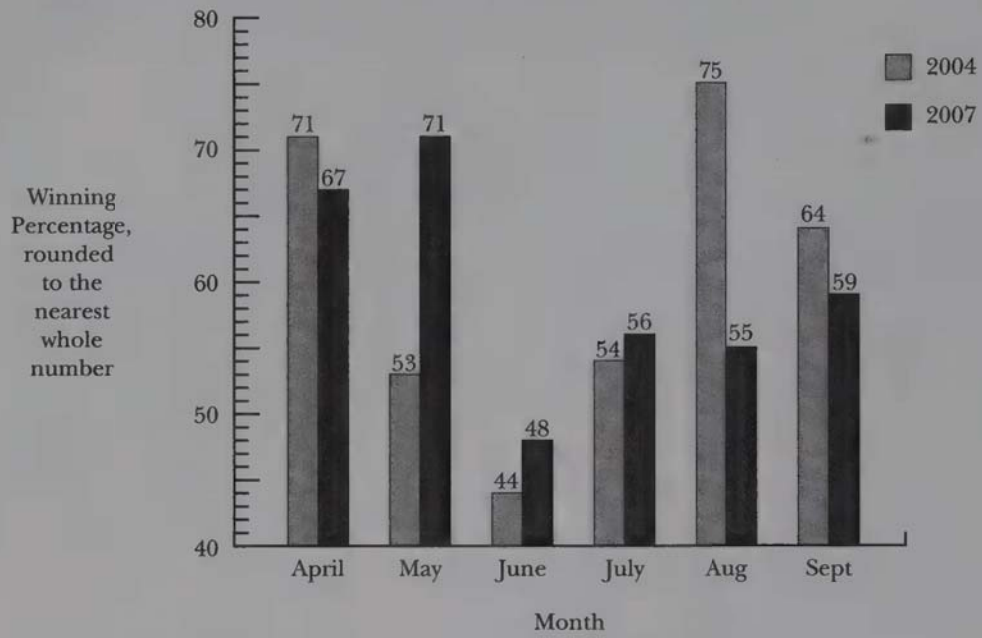
21.3 Identify the months in which the 2007 Red Sox team had a higher winning percentage than the 2004 Red Sox team.

This diagram is called a grouped bar chart because each month along the horizontal axis has two vertical bars—a lighter-colored bar that represents the 2004 winning percentage and a darker-colored bar that represents the 2007 winning percentage. The team had a higher winning percentage in 2007 for each month in which the darker bar is taller than the lighter bar: May, June, and July.

Note: Problems 21.3–21.4 refer to the diagram in Problem 21.3, a grouped bar chart that compares the winning percentages of the Boston Red Sox each month during two years they won the World Series.

21.4 Identify the month in which the 2004 and 2007 teams had the highest combined winning percentage and the month in which the combined winning percentage was lowest.

The heights of the individual columns represent the winning percentages for the 2004 and 2007 teams each month, as illustrated below.

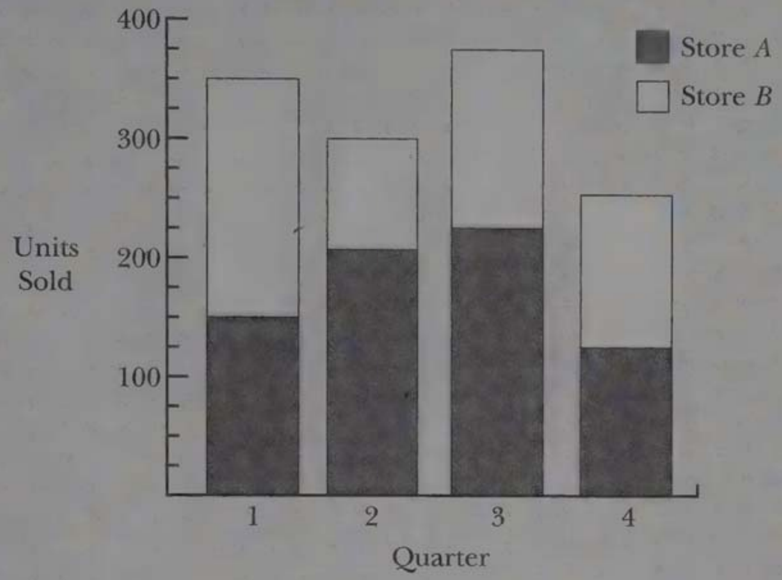


Add the winning percentages for the 2004 and 2007 teams each month. The highest and lowest combined winning percentages are, respectively, the highest and lowest sums in the following chart.

Month	*Total Winning %*
April	$71+67=138$
May	$53+71=124$
June	$44+48=92$
July	$54+56=110$
August	$75+55=130$
September	$64+59=123$

You conclude that the 2004 and 2007 Red Sox teams had the highest combined winning percentage in April and the lowest winning percentage in June.

Note: Problems 21.5–21.7 refer to the stacked column bar chart below, which compares the total sales of an item at two different retail locations during four quarters of a financial year.



21.5 Rank the quarters according to total combined sales at both stores, from least to greatest.

This problem is a lot like Problem 21.4, but because the bars are stacked, it's easier to figure out which combined total is highest—just look for the tallest stacked column.

A stacked column bar chart combines similar data from multiple sources and creates a single column that represents all of those sources at once. In Problems 21.3–21.4, a grouped column bar chart compared data about two different baseball seasons by placing columns representing similar data next to one another. In a stacked bar chart, similar groups of data are combined into a single column.

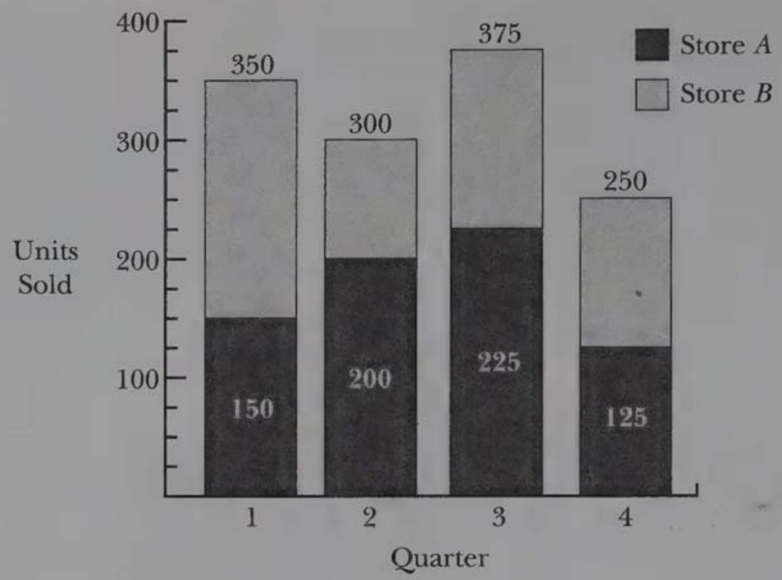
In this chart, the leftmost column reaches a height of 350, so a total of 350 units were sold in Quarter 1. A portion of those units was sold in Store *A* (represented by the darker region at the bottom of the column) and a portion was sold in Store *B* (the lighter region at the top of the column).

The shortest column represents the least total sales, and the tallest column represents the greatest total combined sales. To rank the quarters from least to greatest, list the columns from the shortest to the tallest in the graph: Quarter 4, Quarter 2, Quarter 1, and Quarter 3.

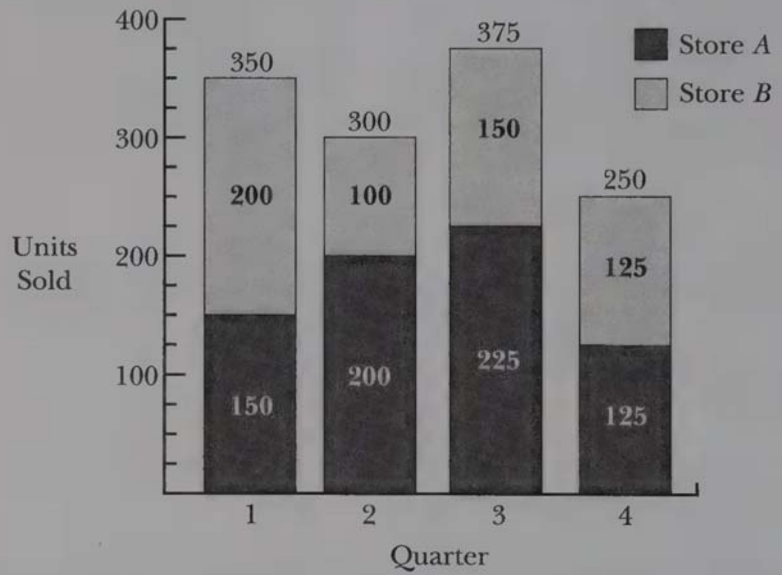
Note: Problems 21.5–21.7 refer to the stacked column bar chart in Problem 21.5, which compares the total sales of an item at two different retail locations during four quarters of a financial year.

21.6 Identify the quarters in which Store *A* sold more units than Store *B*.

Store *A* recorded more sales than Store *B* whenever Store *A*'s sales represent more than half of the total sales that quarter. In the following figure, the height of each stacked column is identified (which represents total combined sales), as well as the height of the darker region at the bottom of each column (which represents sales at Store *A*).



To calculate the total units sold at Store *B*, subtract Store *A*'s sales from the total combined sales. For example, the total combined sales for Quarter 1 is 350, and Store *A* accounts for 150 units sold. Therefore, Store *B* sold 350 – 150 = 200 units. The sales of Store *B* are calculated in the following stacked column bar chart.



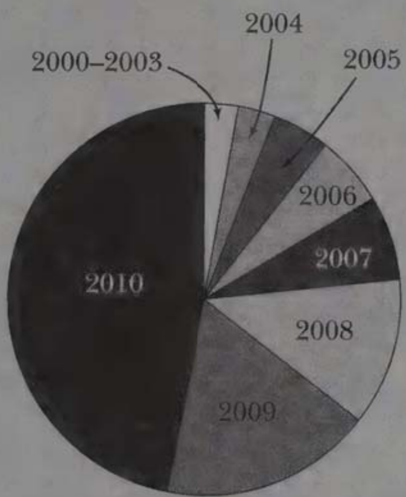
Store *A* sold more units than Store *B* in Quarter 2 (200 > 100) and Quarter 3 (225 > 150). In Quarter 1, Store *B* outsold Store *A*, and in Quarter 4, both stores sold the same number of units.

Note: Problems 21.5–21.7 refer to the stacked column bar chart in Problem 21.5, which compares the total sales of an item at two different retail locations during four quarters of a financial year.

21.7 Rank the quarters according to total sales at Store *B*, from least to greatest.

According to the solution to Problem 21.6, Store *B* sold 200 units in Quarter 1, 100 units in Quarter 2, 150 units in Quarter 3, and 125 units in Quarter 4. Rank the quarters by units sold from least to greatest: Quarter 2, Quarter 4, Quarter 3, and Quarter 1.

21.8 Consider the pie chart below, representing total sales of electronic books (eBooks) by year between 2000 and 2010, based on sales revenue. Describe the general trend in sales, referencing the chart to support your conclusion.

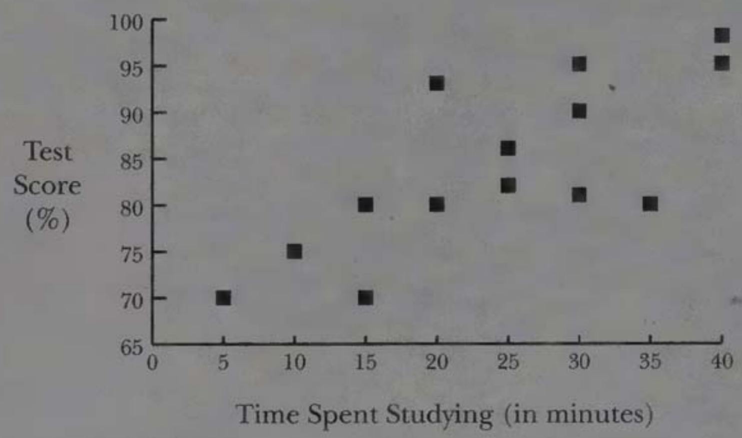


Total eBook sales, 2000–2010

Sectors are pie slice-shaped pieces of the circle. In this graph, each year has its own sector except for 2000–2003, which are grouped into a single sector.

Revenue generated by eBook sales has increased each year since 2004, as the areas of the sectors representing successive years increase. This trend began in 2004, when total sales were approximately equal to eBook sales in the years 2000–2003 combined. Although sales have steadily climbed, the most dramatic jumps in sales occurred during the last three years. More than 75% of total eBook sales occurred between 2008 and 2010, but even more noteworthy was the year 2010, in which nearly half of the sales between 2000 and 2010 occurred.

21.9 The scatter plot below illustrates the relationship between the time individual students studied for a test and their scores on that test. What score did the student who studied 35 minutes receive? Is there a relationship between the length of time studied and scores earned on the test?



Time studied is recorded along the horizontal axis. A single data point appears above the number 35; that point appears at a height of 80 along the vertical axis. Therefore, the student who studied 35 minutes earned a score of 80%.

There appears to be a relationship between the time studied and the scores earned. Statistically, this means that a *correlation* exists between the data sets. Generally speaking, the further left the points are on the scatter plot, the lower they are on the graph. In other words, the less time a student spent studying, the less likely it is that the student received a high grade. Similarly, points nearer the right side of the graph are generally higher, so longer periods of study generally led to higher scores.

There are a few exceptions to the trend. One student who studied 20 minutes earned a 93%, while another student who studied the same length of time earned an 80%. The student who studied 35 minutes scored the same as two students who studied significantly less. These exceptions do not nullify the positive correlation between studying and scores. Real-life data is rarely perfect—you should not expect the student scores to lie on a perfectly straight line.

Stem-and-Leaf Plots

Data that looks like bar charts

Note: Problems 21.10–21.12 refer to the stem-and-leaf plot below, which reports the total number of guests at a hotel each day over a two-week period, such that 4|8 = 48.

4	8			
5	2	4		
6	1	1	3	6
7	3	6	8	
8	0	5	7	9

21.10 Transform the leaves of the stem value 8 into data values.

A stem-and-leaf plot is a visual organization tool that groups data according to a shared *stem*. In this diagram, the stems are 4, 5, 6, 7, and 8; they are listed in a column left of the vertical line. Notice that the problem states 4|8 = 48. Therefore, each stem represents the tens digit of a number, and the leaves listed to the right of the stems are ones digits.

The stem value 8 has four leaves, which appear to the right of the vertical line separating the stems from the leaves: 0, 5, 7, and 9. Place the digit 8 in front of each leaf value to convert the bottom row of the stem-and-leaf plot into data values: 80, 85, 87, and 89.

Note: Problems 21.10–21.12 refer to the stem-and-leaf plot in Problem 21.10, which reports the total number of guests at a hotel each day over a two-week period, such that 4|8 = 48.

21.11 On exactly two days during the two-week period, the hotel accommodated the same number of guests. What was the total number of registered guests on each of those two days?

Equal values in a stem-and-leaf plot must occur in the same row—they must share the same stem. In this diagram, the row with a stem of 6 has two leaves that equal 1. Therefore, on two days during the two-week period, there were 6|1 = 61 guests in the hotel.

Note: Problems 21.10–21.12 refer to the stem-and-leaf plot in Problem 21.10, which reports the total number of guests at a hotel each day over a two-week period, such that 4|8 = 48.

21.12 How many days did the hotel accommodate between 70 and 79 guests during the two-week period? What percentage of the total number of guests does this represent (rounded to the nearest hundredth)?

The stem value 7 has three leaves: 3, 6, and 8. Therefore, three of the days during the two-week period featured guest totals of 73, 76, and 78, for a total of 227 guests. To determine the percentage of total guests this represents, you need to convert all of the stem-and-leaf data to actual values and calculate the total.

$$\begin{array}{r|llll}
4 & 8 & & & \\
5 & 2 & 4 & & \\
6 & 1 & 1 & 3 & 6 \\
7 & 3 & 6 & 8 & \\
8 & 0 & 5 & 7 & 9
\end{array} = \begin{array}{l}
48 \\
52, 54 \\
61, 61, 63, 66 \\
73, 76, 78 \\
80, 85, 87, 89
\end{array}$$

Calculate the sum of the data values to compute the total number of guests.

$$48 + 52 + 54 + 61 + 61 + 63 + 66 + 73 + 76 + 78 + 80 + 85 + 87 + 89 = 973$$

Divide the number of guests that stayed when there were between 70 and 79 registered guests by the total number of guests.

$$\begin{aligned}
&\frac{\text{total when number of guests is in the 70s}}{\text{total number of guests during two-week period}} \\
&= \frac{227}{973} \\
&\approx 0.2332991 \\
&\approx 23.33\%
\end{aligned}$$

Approximately 23.33% of the total guests during the two-week period stayed at the hotel on the three days when guest registration was between 70 and 79.

21.13 A sample of smart phone customers is asked how many applications (or "apps") they paid for in the last year. Their responses appear in the ordered list below. Construct a stem-and-leaf plot representing their responses.

1, 5, 7, 8, 9, 11, 13, 15, 15, 18, 20, 22, 26, 29

The stem of a data set is often the tens digit, and that is true for this set of data. Separate the values according to their tens digits.

$$\begin{array}{ll}
\text{0 tens digit:} & 1, 5, 7, 8, 9 \\
\text{1 tens digit:} & 11, 13, 15, 15, 18 \\
\text{2 tens digit:} & 20, 22, 26, 29
\end{array}$$

The tens digit of each row is the stem value for that row. The ones values in each row are the leaves. For example, there are five data values with a digit of 1 in the tens place: 11, 13, 15, 15, and 18. The common tens digit 1 is the stem and the corresponding leaves are 1, 3, 5, 5, and 8.

$$\begin{array}{r|lllll}
0 & 1 & 5 & 7 & 8 & 9 \\
1 & 1 & 3 & 5 & 5 & 8 \\
2 & 0 & 2 & 6 & 9 &
\end{array} \qquad \text{Note: } 2|0 = 20$$

21.14 A small pet store records the number of fish purchased each month for one year; the data is compiled below. Construct a stem-and-leaf plot for total fish sales by month.

95, 98, 114, 119, 128, 110, 98, 79, 100, 120, 93, 112

Begin by writing the values in order, from least to greatest.

79, 93, 95, 98, 98, 100, 110, 112, 114, 119, 120, 128

This data set has a minimum of 79 and a maximum of 128, so the stems will range from 7 to 12. Note that three of the stems (10, 11, and 12) include hundreds digits as well as tens digits. Once again, only the ones digits appear as leaves.

7	9
8	
9	3 5 8 8
10	0
11	0 2 4 9
12	0 8

Note: $10|0 = 100$

This stem has no leaves (there are no values between 80 and 89) but you should include it and leave it blank.

21.15 Construct a stem-and-leaf plot for the data set below, the number of tackles made by each member of the NFL Pittsburgh Steeler defense during the 2010 season.

132	109	100	90	82	63	59	50	39	33
22	20	20	18	16	15	12	7	7	4
3	2	2	2	0					

The stems for this set of data range between 0 and 13, as the smallest values have no digit in the tens place and the largest value has a hundreds digit of 1 and a tens digit of 3.

0	0 2 2 2 3 4 7 7
1	2 5 6 8
2	0 0 2
3	3 9
4	
5	0 9
6	3
7	
8	2
9	0
10	0 9
11	
12	
13	2

Note: $13|2 = 132$

Measures of Central Tendency

Mean, median, and mode

Note: Problems 21.16–21.19 refer to the data set below, student percentage scores on a midterm exam.

52, 63, 68, 76, 79, 80, 83, 85, 88, 91, 91, 97, 100

21.16 Calculate the average (arithmetic mean) of the data.

Some books omit the word "arithmetic."

Note that the problem specifies the arithmetic mean—this is the calculation most familiar to students, commonly called the *average.* However, the word "average" is not specific enough; it describes the "middle" of the data, but the arithmetic mean does not always return the most correct answer to the question "What is the central point in the data set?"

The arithmetic mean of a data set, written $\bar{x}$, is equal to the sum of the data values divided by n, the number of data values.

You read this as "x bar."

$$\begin{aligned}\bar{x} &= \frac{\text{sum of data}}{n} \\ &= \frac{52+63+68+76+79+80+83+85+88+91+91+97+100}{13} \\ &= \frac{1{,}053}{13} \\ &= 81\end{aligned}$$

The arithmetic mean of the data is a test score of 81.

Note: Problems 21.16–21.19 refer to the data set below, student percentage scores on a midterm exam.

52, 63, 68, 76, 79, 80, 83, 85, 88, 91, 91, 97, 100

21.17 Calculate the median of the data.

The median of a data set is the middle value when the data is ordered from least to greatest. Just as a grassy median splits a highway into two equal halves, the statistical median splits a data set into two equal halves. This set of data contains $n = 13$ values, so the median is the seventh value in the ordered list, with six values preceding it and six values following it.

$$52, 63, 68, 76, 79, 80, \underline{83}, 85, 88, 91, 91, 97, 100$$

The median is underlined in the above list: 83. Note that six values precede it (52, 63, 68, 76, 79, and 80) and six values follow it (85, 88, 91, 91, 97, and 100). When a data set contains an odd number of values (like this data set, in which $n = 13$), the median is the $\left(\frac{n+1}{2}\right)$th value in the ordered list. Substitute $n = 13$ into the formula to verify this claim.

$$\begin{aligned}\frac{n+1}{2} &= \frac{13+1}{2}\\ &= \frac{14}{2}\\ &= 7\end{aligned}$$

According to these calculations, the median is the seventh value in the ordered data set, the number 83. Remember that this formula is only true when the data set has an odd number of values.

Note: Problems 21.16–21.19 refer to the data set below, student percentage scores on a midterm exam.

52, 63, 68, 76, 79, 80, 83, 85, 88, 91, 91, 97, 100

21.18 Calculate the mode of the data.

The mode of a data set is the value that appears most often. Two students scored 91% on the test, and that is the only score that repeats, so the mode of the test scores is 91.

If the data contains no repeating values, there is no mode.

Note: Problems 21.16–21.19 refer to the data set below, student percentage scores on a midterm exam.

52, 63, 68, 76, 79, 80, 83, 85, 88, 91, 91, 97, 100

21.19 A student absent on the day the test was administered returns to school. Once her score is added to the data set, the class arithmetic mean is exactly equal to 82. What score did she earn on the test?

To compute the arithmetic mean, add the total scores of the students and divide by the total number of test scores. According to Problem 21.16, $n = 13$ students took the test and the sum of their scores was 1,053 before the new score—described in this problem—is added. Let x represent the new student's score. In order to calculate the new class score average, you will need to add x to the total sum of the scores and divide by $n = 14$. This problem states that the new average is $\bar{x} = 82$.

You are including one extra student in the average now, so you need to add her score to the total (you don't know what her score is, so you use the variable x) and divide by the total number of students, now 14 instead of 13.

$$\begin{aligned}\bar{x} &= \frac{\text{sum of scores}}{n}\\ 82 &= \frac{1{,}053+x}{14}\end{aligned}$$

Cross-multiply to solve the proportion for x.

$$\begin{aligned}\frac{82}{1} &= \frac{1{,}053+x}{14}\\ 82(14) &= 1(1{,}053+x)\\ 1{,}148 &= 1{,}053+x\\ 1{,}148-1{,}053 &= x\\ 95 &= x\end{aligned}$$

The final student to complete the exam scored 95%, which increased the class average from 81% to 82%.

Note: Problems 21.20–21.23 refer to the data set below, the total calories in 12 popular menu choices at the McDonald's fast food restaurant chain.

440, 510, 740, 540, 460, 790, 750, 530, 380, 360, 500, 530

21.20 Calculate the mode of the data.

Before you perform any statistical analysis on the data, reorder the values from least to greatest. This will help you identify the mode and will assist in the identification of the median, in Problem 21.22.

360, 380, 440, 460, 500, 510, 530, 530, 540, 740, 750, 790

Only one value repeats in the data set: 530. Therefore, the mode for this selection of menu items is 530 calories.

Note: Problems 21.20–21.23 refer to the data set below, the total calories in 12 popular menu choices at the McDonald's fast food restaurant chain.

440, 510, 740, 540, 460, 790, 750, 530, 380, 360, 500, 530

21.21 Calculate the arithmetic mean of the data.

Calculate the sum of the data values and divide by the total number of items for which you are given calorie information: $n = 12$.

$$\begin{aligned}\bar{x} &= \frac{\text{sum of data}}{n}\\ &= \frac{440+510+740+540+460+790+750+530+380+360+500+530}{12}\\ &= \frac{6{,}530}{12}\\ &= 544.1\overline{6}\end{aligned}$$

The 12 menu items have an arithmetic mean of 544.1666666..., or $544.1\overline{6}$, calories.

Note: Problems 21.20–21.23 refer to the data set below, the total calories in 12 popular menu choices at the McDonald's fast food restaurant chain.

440, 510, 740, 540, 460, 790, 750, 530, 380, 360, 500, 530

21.22 Calculate the median of the data.

Place the calorie values in order, from least to greatest, as demonstrated in Problem 21.20.

360, 380, 440, 460, 500, 510, 530, 530, 540, 740, 750, 790

The median is the "middle number" of the ordered data set, which splits the set into an equal number of values that precede and follow the median. This data set contains an even number of values, so unlike Problem 21.17, there is no single value that divides the data evenly. Instead, you select the *two* values in the middle of the data set, 510 and 530.

360, 380, 440, 460, 500, 510, 530, 530, 540, 740, 750, 790

Notice that five values precede the underlined values above and five values follow them as well. To calculate the median, compute the arithmetic mean of the two underlined values.

Add 510 and 530. Then divide by 2.

$$\begin{aligned}\text{median} &= \frac{510+530}{2}\\ &= \frac{1{,}040}{2}\\ &= 520\end{aligned}$$

The median of the data set is 520 calories. Note that the median is not always one of the data values in the set—none of the menu items contains 520 calories. This should not be off-putting, as you do not assume that another common measure of central tendency—the arithmetic mean—is necessarily a data value either; in this example, no item contains $\bar{x} = 544.1\overline{6}$ calories.

If a data set contains an even number of values, the formula $\frac{n}{2}$ identifies the first of the two numbers for which the arithmetic mean is the median of the data set. In this problem, $n = 12$, so the arithmetic mean of the data value in place $\frac{12}{2} = 6$ (510 calories) and the value that immediately follows it (530 calories, in position seven) is the median of the data set.

Note: Problems 21.20–21.23 refer to the data set below, the total calories in 12 popular menu choices at the McDonald's fast food restaurant chain.

440, 510, 740, 540, 460, 790, 750, 530, 380, 360, 500, 530

21.23 Armed with nutrition information about the menu items, you choose to avoid the three items that contain the most calories. Calculate the arithmetic mean of the smaller data set that excludes those three items.

Eliminating the three highest calorie items (containing 740, 750, and 790 calories) and ordering the remaining values from least to greatest produces the following data set.

360, 380, 440, 460, 500, 510, 530, 530, 540

To compute the average, add the values and divide by the total number of items.

$$\begin{aligned}\bar{x} &= \frac{\text{sum of data}}{n}\\ &= \frac{360+380+440+460+500+510+530+530+540}{9}\\ &= \frac{4{,}250}{9}\\ &= 472.\overline{2}\end{aligned}$$

The remaining nine menu items have an arithmetic mean of $472.\overline{2}$ calories.

Note: Problems 21.24–21.25 refer to the data set below, homeruns hit each year by Babe Ruth and Roger Maris during their careers. Note that Babe Ruth's homerun totals are listed beginning in 1918, the year in which the number of games he played per season significantly increased.

Ruth	*11*	*29*	*54*	*59*	*35*	*41*	*46*	*25*	*47*
	60	*54*	*46*	*49*	*46*	*41*	*34*	*22*	*6*
Maris	*14*	*28*	*9*	*19*	*16*	*39*	*61*	*33*	*23*
	26	*8*	*13*	*9*	*5*				

21.24 Before the steroid era of baseball, the record-breaking mark of 61 homeruns in a year set by Roger Maris—narrowly edging out the longstanding record of 60 homeruns set by Babe Ruth—stood for decades. Compute the mean and median number of homeruns per year for Babe Ruth, rounding each to the nearest whole number.

Calculate the sum of Babe Ruth's homeruns per year and divide by the total number of years represented, $n = 18$.

$$\begin{aligned}\bar{x} &= \frac{\text{sum of data}}{n}\\ &= \frac{705}{18}\\ &= 39.1\bar{6}\\ &\approx 39\end{aligned}$$

You could write the values from greatest to least instead—as long as they are sequential.

To compute the median, you should first rewrite the data values in order, from least to greatest.

6, 11, 22, 25, 29, 34, 35, 41, 41, 46, 46, 46, 47, 49, 54, 54, 59, 60

There are $n = 18$ data values—an even number—so the value in position $\frac{n}{2}$ is the first of the two consecutive values in the ordered data set that must be averaged to calculate the median.

$$\begin{aligned}\frac{n}{2} &= \frac{18}{2}\\ &= 9\end{aligned}$$

The data value in position nine is 41, and the data value in position ten is 46. Compute the arithmetic mean.

$$\begin{aligned}\text{median} &= \frac{41+46}{2}\\ &= \frac{87}{2}\\ &= 43.5\\ &\approx 44\end{aligned}$$

Beginning in 1918 and throughout the rest of his career, Babe Ruth hit an arithmetic mean of 39 and a median of 44 homeruns per season.

Note: Problems 21.24–21.25 refer to the data set below, homeruns hit each year by Babe Ruth and Roger Maris during their careers. Note that Babe Ruth's homerun totals are listed beginning in 1918, the year in which the number of games he played per season significantly increased.

Ruth	*11*	*29*	*54*	*59*	*35*	*41*	*46*	*25*	*47*
	60	*54*	*46*	*49*	*46*	*41*	*34*	*22*	*6*
Maris	*14*	*28*	*9*	*19*	*16*	*39*	*61*	*33*	*23*
	26	*8*	*13*	*9*	*5*				

21.25 Compute the mean and median number of homeruns per year for Roger Maris, rounding each to the nearest whole number. Based on your answers to Problems 21.24–21.25, who was the better homerun hitter?

The arithmetic mean for Roger Maris is his total number of homeruns divided by $n = 14$ seasons.

$$\begin{aligned}\bar{x} &= \frac{\text{sum of data}}{n}\\ &= \frac{303}{14}\\ &= 21.6\overline{428571}\\ &\approx 22\end{aligned}$$

To calculate the median, first list Maris' homeruns in order, from least to greatest.

5, 8, 9, 9, 13, 14, 16, 19, 23, 26, 28, 33, 39, 61

The data set contains an even number of values, so apply the formula $\frac{n}{2}$ to identify the position of the first of two consecutive data values used to calculate the median.

$$\begin{aligned}\frac{n}{2} &= \frac{14}{2}\\ &= 7\end{aligned}$$

The seventh and eighth data values in the ordered set are 16 and 19. Their arithmetic mean is the median for the data set.

$$\begin{aligned}\text{median} &= \frac{16+19}{2}\\ &= \frac{35}{2}\\ &= 17.5\\ &\approx 18\end{aligned}$$

You conclude that Roger Maris hit an arithmetic mean of 22 and a median of 18 homeruns per season, significantly lower than the yearly statistics for Babe Ruth. In fact, in 12 of his 14 seasons, Maris's homerun total was less than Ruth's lifetime average.

If you compare their performances using the arithmetic means, Maris hit $39 - 22 = 17$ fewer homeruns per year on average. Using the medians confirms

this disparity: Maris hit 44 – 18 = 26 fewer homeruns per year. Maris's single-season homerun record was as significant as it was surprising statistically. Clearly, Ruth was the more consistently successful homerun hitter, but a single outstanding season by Maris eclipsed his record of 60 homeruns.

The value of this historical statistical study is somewhat lessened by the ubiquitous use of steroids that began at the end of the twentieth century, artificially inflating statistics and making it difficult, if not impossible, to reconcile the data from the two different eras.

Quartiles

Marking the 25% and 75% points in a data set

Problems 21.26–21.30 refer to the data set below, the number of pets owned by members of an elementary school class.

0, 1, 1, 2, 3, 4, 4, 5, 6

21.26 Calculate the second quartile (Q_2) for the data set.

Quartiles divide a data set into four quarters, each representing approximately 25% of the data. Not all statisticians agree upon a singular method to accomplish this task, so different textbooks present different techniques. No particular method is inherently more accurate than any other, but the methods described in this section represent the most common techniques for calculating quartiles in a basic statistics class.

This weird looking thing is just an italicized Q.

The first quartile Q_1 (or *lower quartile*) denotes the endpoint for the first 25% of the data, and the third quartile (or *upper quartile*) marks the 75% boundary for the data. If you have completed the preceding problems in this section, you are already familiar with the second quartile Q_2, which marks the halfway point in the data—also called the median. Apply the techniques from the previous section to calculate the median for this data set, which has an odd number of values.

Look at Problems 21.17 and 21.22–21.25 to review calculating the median.

$$\begin{aligned}\frac{n+1}{2} &= \frac{9+1}{2}\\ &= \frac{10}{2}\\ &= 5\end{aligned}$$

The value in the fifth position of the ordered data set is the median: 3.

Problems 21.26–21.30 refer to the data set below, the number of pets owned by members of an elementary school class.

$$0, 1, 1, 2, 3, 4, 4, 5, 6$$

21.27 Calculate the first quartile (Q_1) for the data set.

According to Problem 21.26, the median of the data set is 3 pets. This value divides the data into two equal halves. The lower half contains values 0, 1, 1, and 2; the upper half contains values 4, 4, 5, and 6.

0, 1, 1, 2	3	4, 4, 5, 6
lower half	median	upper half

The first quartile, Q_1, is the median of the lower half of the data values 0, 1, 1, and 2. To calculate the median of a data set containing $n = 4$ values, compute the arithmetic mean of the middle two values (in this case, 1 and 1). You conclude that $Q_1 = 1$.

The average of any number and itself (like 1 and 1) is that number: 1.

Problems 21.26–21.30 refer to the data set below, the number of pets owned by members of an elementary school class.

$$0, 1, 1, 2, 3, 4, 4, 5, 6$$

21.28 Calculate the third quartile (Q_3) for the data set.

As Problem 21.27 demonstrates, the median of the data (3) separates the ordered values into two equal halves.

0, 1, 1, 2	3	4, 4, 5, 6
lower half	median	upper half

The third quartile, Q_3, is the median of the upper half of the data: 4, 4, 5, and 6. Like Problem 21.27, this subset of the data contains an even number of values: $n = 4$. Thus, the median is equal to the average of the two middle values, 4 and 5.

$$\begin{aligned} Q_3 &= \frac{4+5}{2} \\ &= \frac{9}{2} \\ &= 4.5 \end{aligned}$$

You conclude that $Q_3 = 4.5$.

Problems 21.26–21.30 refer to the data set below, the number of pets owned by members of an elementary school class.

0, 1, 1, 2, 3, 4, 4, 5, 6

21.29 Generate a five-number summary for the data.

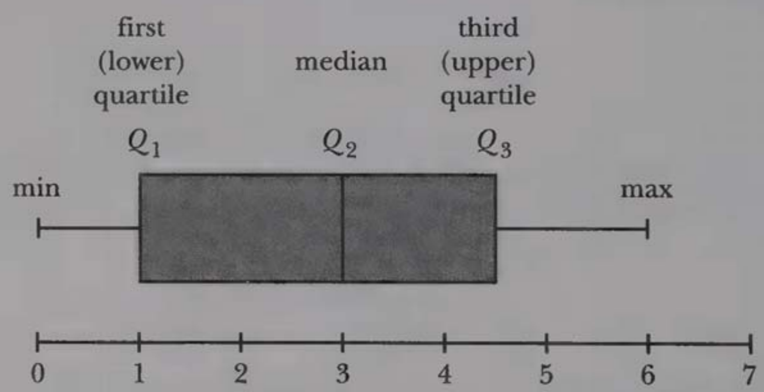
A *five-number summary* describes a set of data by listing, in order, the following values: the minimum (the smallest value in the data set), the lower quartile (Q_1), the median (Q_2), the upper quartile (Q_3), and the maximum. This data set has a minimum value of 0 and a maximum value of 6; according to Problems 21.26–21.28, $Q_1 = 1$, $Q_2 = 3$, and $Q_3 = 4.5$. Therefore, the five-number summary for this data set is 0, 1, 3, 4.5, 6.

Problems 21.26–21.30 refer to the data set below, the number of pets owned by members of an elementary school class.

0, 1, 1, 2, 3, 4, 4, 5, 6

21.30 Construct a box-and-whisker plot to represent the data.

A box-and-whisker plot is a visual representation of the five-number summary, as described in Problem 21.29. It consists of two vertical line segments called the *whiskers* (which represent the minimum and maximum values of the data) and a rectangular *box* whose left and right boundaries are the lower and upper quartiles. The median is drawn as a line segment between and parallel to the segments representing Q_1 and Q_3.



Problems 21.31–21.34 refer to the data set below, the length of time (in hours) a sample of adults spent reading for pleasure during a one-week period.

10, 13, 2, 5, 31, 16, 1, 20, 2, 12

21.31 Calculate Q_2 for the data set.

As Problem 21.26 explains, the second quartile (Q_2) is the median of the data set. However, in order to identify the median, the data values must be listed in order, from least to greatest.

1, 2, 2, 5, 10, 12, 13, 16, 20, 31

There are $n = 10$ data values, an even number, so the median is equal to the arithmetic mean of the number in position $\frac{n}{2} = \frac{10}{2} = 5$ and the number that follows it: 10 and 12.

$$\begin{aligned} Q_2 &= \frac{10+12}{2} \\ &= \frac{22}{2} \\ &= 11 \end{aligned}$$

The second quartile of the data set is $Q_2 = 11$ hours of pleasure reading.

Problems 21.31–21.34 refer to the data set below, the length of time (in hours) a sample of adults spent reading for pleasure during a one-week period.

10, 13, 2, 5, 31, 16, 1, 20, 2, 12

21.32 Calculate the lower quartile (Q_1) for the data set.

According to Problem 21.31, the median of the ordered data set below is $Q_2 = 11$.

1, 2, 2, 5, 10, 12, 13, 16, 20, 31

The median splits the data into two equal halves. Note that the median does not actually appear within the data set, so all of the data values belong either to the lower or upper half of the data.

1, 2, 2, 5, 10	11	12, 13, 16, 20, 31
lower half	median	upper half

The lower quartile is the median of the lower half of the data, which contains the values 1, 2, 2, 5, and 10. The middle value of this set is the number in the third position: $Q_1 = 2$.

Unlike Problems 21.26–21.28, when the median 3 was a data value. It split the data into two halves but wasn't contained in either. Remember, some books do this a little differently, because for some reason, statisticians can't agree on one way to calculate quartiles. Weird.

Problems 21.31–21.34 refer to the data set below, the length of time (in hours) a sample of adults spent reading for pleasure during a one-week period.

10, 13, 2, 5, 31, 16, 1, 20, 2, 12

21.33 Calculate the upper quartile (Q_3) for the data set.

According to Problem 21.32, the upper half of the data set is 12, 13, 16, 20, and 31. The upper quartile is the median of this ordered set containing $n = 5$ values: $Q_3 = 16$.

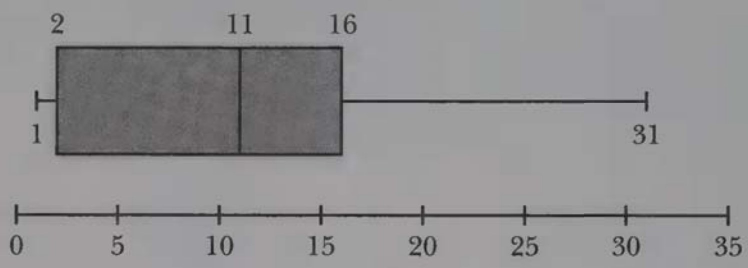
Problems 21.31–21.34 refer to the data set below, the length of time (in hours) a sample of adults spent reading for pleasure during a one-week period.

10, 13, 2, 5, 31, 16, 1, 20, 2, 12

21.34 Generate a five-number summary for the data and use it to construct a box-and-whisker plot.

The minimum value of the data set is 1 and the maximum value is 31. According to Problems 21.31–21.33, $Q_1 = 2$, $Q_2 = 11$, and $Q_3 = 16$. Therefore, the five-number summary for the data is 1, 2, 11, 16, 31.

The box-and-whisker plot for the data illustrates the maximum and minimum values as small vertical segments that extend toward a rectangle bounded on the left by Q_1, bounded on the right by Q_3, and divided by a vertical segment located at Q_2, as illustrated below.



Note: Problems 21.35–21.37 refer to data set below, the monthly charges (in dollars) a sample of consumers pay for monthly cable or satellite television service.

74, 82, 128, 75, 45, 61, 53, 107, 78, 65, 96

A sample is a small, representative group from a population, because asking a giant population of people something isn't really feasible.

21.35 Calculate the median monthly charge.

List the monthly charges in order, from least to greatest.

45, 53, 61, 65, 74, 75, 78, 82, 96, 107, 128

The median of an ordered set containing an odd number of data values ($n = 11$) is the number in position six, as calculated below.

$$\begin{aligned}\frac{n+1}{2} &= \frac{11+1}{2} \\ &= \frac{12}{2} \\ &= 6\end{aligned}$$

The sixth value in the ordered list is 75, so the median price paid for monthly cable or satellite television service is \$75.

Note: Problems 21.35–21.37 refer to data set below, the monthly charges (in dollars) a sample of consumers pay for monthly cable or satellite television service.

74, 82, 128, 75, 45, 61, 53, 107, 78, 65, 96

21.36 Calculate the lower and upper quartiles for the data.

According to Problem 21.35, the median $Q_2 = 75$ divides the ordered data into two halves, each containing five data values. The median is a value in the data set, so it is excluded from both halves of the data.

45, 53, 61, 65, 74	75	78, 82, 96, 107, 128
lower half	median	upper half

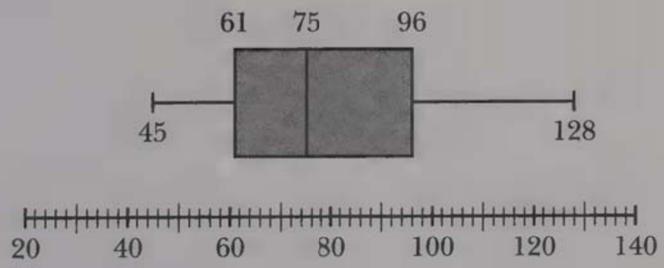
The lower half of the data contains the values 45, 53, 61, 65, and 74; the third number of this ordered data set is the median, and is therefore the lower quartile: $Q_1 = 61$. The upper half of the data contains the values 78, 82, 96, 107, and 128. Once again, the third number of this data set represents a quartile: $Q_3 = 96$.

Note: Problems 21.35–21.37 refer to data set below, the monthly charges (in dollars) a sample of consumers pay for monthly cable or satellite television service.

74, 82, 128, 75, 45, 61, 53, 107, 78, 65, 96

21.37 Generate a five-number summary for the data and use it to construct a box-and-whisker plot.

According to Problems 21.35–21.36, $Q_1 = 61$, $Q_2 = 75$, and $Q_3 = 96$. The maximum and minimum values are 45 and 128, respectively. Therefore, the five-number summary for the data is 45, 61, 75, 96, 128. Apply these values to construct the following box-and-whisker plot.



Appendix A

Multiplication Table

The product of the numbers in the top row and the left column is the value at which that row and column intersect.

	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Appendix B
Important Formulas

Chapter 6

Divisibility Tests for Integers

2: Even numbers

3: Sum of digits is divisible by 3

4: Last two digits divisible by 2

5: Ones digit is 0 or 5

6: Divisible by 2 and divisible by 3

7: Separate ones digit, double it, and subtract it from original number excluding the ones digit; test divisibility of difference by 7

8: Last three digits divisible by 8

9: Sum of digits divisible by 9

10: Ones digit is 0

Chapter 7

* Improper fraction to mixed number:
  If $A \div B$ has quotient Q and remainder R, then $\frac{A}{B} = Q\frac{R}{B}$
* Mixed number to improper fraction: $W\frac{X}{Y} = \frac{WY + X}{Y}$

Chapter 9

* Multiplying fractions: $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$
* Dividing fractions: $\frac{w}{x} \div \frac{y}{z} = \frac{wz}{xy}$

Chapter 12

* Percentage increase and decrease: $s \cdot p = c$
* Cross-multiplication: $\frac{a}{b} = \frac{c}{d}$ is equivalent to $ad = bc$
* Percentage proportion formula: $\frac{c}{100} = \frac{p}{w}$

Chapter 13

Exponential Rules:

* $(a^x)(a^y) = a^{x+y}$
* $\frac{b^v}{b^w} = b^{v-w}$
* $(c^x)^y = c^{xy}$
* $\left(\frac{p}{q}\right)^{-1} = \frac{q}{p}$

Chapter 14

* Multiplying radicals: $\sqrt{a}\left(\sqrt{b}\right) = \sqrt{ab}$
* Dividing radicals: $\sqrt{x} \div \sqrt{y} = \sqrt{\frac{x}{y}}$
* Fractional powers: $x^{R/S} = \sqrt[S]{x^R}$

  or

  $x^{R/S} = \left(\sqrt[S]{x}\right)^R$

Chapter 16

* FOIL Method: $(a+b)(c+d) = ac + ad + bc + bd$
* Multiplying expressions: $(a+b)(x+y+z) = ax + ay + az + bx + by + bz$

Chapter 18

* Slope of the line through points (x_1, y_1) and (x_2, y_2): $\dfrac{y_2 - y_1}{x_2 - x_1}$

Chapter 19

* Pythagorean Theorem: $a^2 + b^2 = c^2$

AREA FORMULAS

* Area of a triangle: $A = \frac{1}{2}bh$
* Area of a rectangle: $A = lw$
* Area of a rhombus: $A = sh$ or $A = \frac{1}{2}(d_1)(d_2)$
* Area of a square: $A = s^2$
* Area of a parallelogram: $A = bh$
* Area of a trapezoid: $A = \frac{h}{2}(b_1 + b_2)$
* Area of a circle: $A = \pi r^2$

PERIMETER FORMULAS

* Perimeter of a rectangle: $P = 2l + 2w$
* Perimeter of a rhombus/square: $P = 4s$
* Circumference of a circle: $C = 2\pi r$

VOLUME FORMULAS

* Volume of a prism: $V = A_B h$
* Volume of a rectangular prism: $V = l \cdot w \cdot h$
* Volume of a cylinder: $V = \pi r^2 h$
* Volume of a pyramid: $V = \frac{1}{3} A_B h$
* Volume of a cone: $V = \frac{1}{3}\pi r^2 h$
* Volume of a sphere: $V = \frac{4}{3}\pi r^3$

Chapter 21

* arithmetic mean: $\bar{x} = \dfrac{\text{sum of data}}{\text{number of data values}}$

Appendix C
Important Concepts

Chapter 2

* Associative Property of Addition: $(a + b) + c = a + (b + c)$
* Associative Property of Multiplication: $(xy)z = x(yz)$
* Commutative Property of Addition: $r + s = s + r$
* Commutative Property of Multiplication: $xy = yx$
* Additive Inverse Property: $x + (-x) = -x + x = 0$
* Multiplicative Inverse Property: $y \cdot \frac{1}{y} = 1$, if $y \neq 0$

Chapter 3

Double Signs

$+ + = +$

$+ - = -$

$- + = -$

$- - = +$

Chapter 4

* Order of Operations:
  1. Parentheses (or other grouping symbols)
  2. Exponents
  3. Multiplication and Division (from left to right)
  4. Addition and Subtraction (from left to right)
* Distributive Property: $a(x + y) = ax + ay$

Chapter 6

* Fundamental Theorem of Arithmetic:
  Every positive integer is a unique product of prime numbers
* Least Common Multiple (LCM):
  The smallest multiple that a set of values have in common
* Greatest Common Factor (GCF):
  The largest number by which a set of values is divisible

Chapter 8

Least Common Denominator (LCD):
The least common multiple of a set of denominators

Chapter 12

* Express percentages as decimals: Move decimal point two places to the left
* Express decimals as percentages: Move decimal point two places to the right

---

Chapter 16

* Like Terms:
  Terms that contain the same variable(s) raised to the same exponent(s)

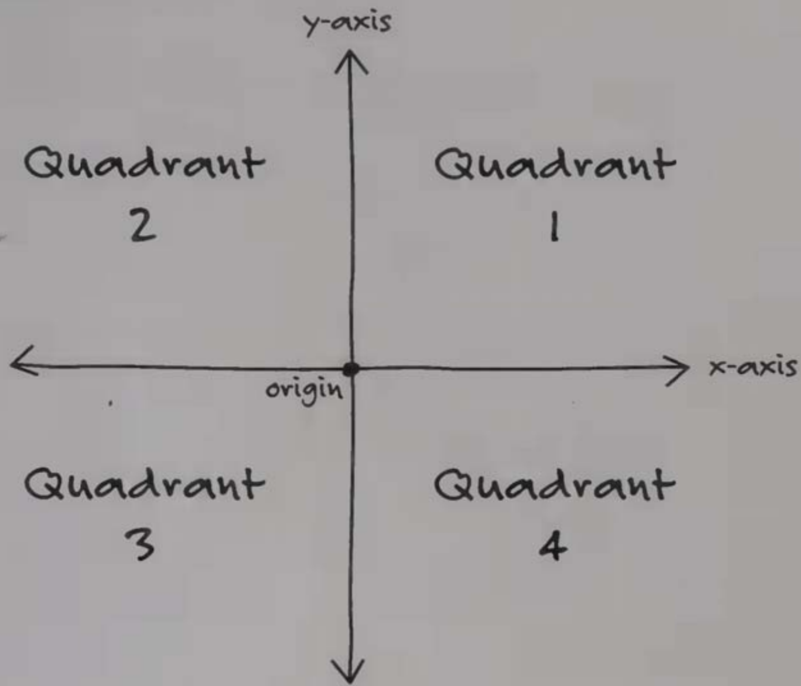
Chapter 17

Steps to Solving Equations

1. Simplify expressions on both sides of the equation independently
2. Move x-terms to one side of the equation
3. Move all other terms to the other side of the equation
4. Combine like terms
5. Eliminate coefficient of x-term

Chapter 18

The Cartesian Coordinate System



Chapter 19

* Right angles: Measure $90°$
* Acute angles: Have measures greater than $0°$ and less than $90°$
* Obtuse angles: have measures greater than $90°$ and less than $180°$

Chapter 21

* Median: Middle value of an ordered data set
* Mode: Data value that occurs most often
* Five-Number Summary: Lists five statistics of a data set in the following order: minimum, Q_1, median, Q_3, maximum.

Appendix D
Measurement Conversion Formulas

ENGLISH UNITS

DISTANCE

* 1 foot = 12 inches
* 1 yard = 3 feet
* 1 mile = 5,280 feet

VOLUME

* 1 cup = 8 fluid ounces
* 1 pint = 2 cups
* 1 quart = 2 pints
* 1 gallon = 4 quarts

WEIGHT

* 1 pound = 16 ounces
* 1 ton = 2,000 pounds

METRIC PREFIXES

* Milli: $\frac{1}{1,000} = 10^{-3}$
* Centi: $\frac{1}{100} = 10^{-2}$
* Deci: $\frac{1}{10} = 10^{-1}$
* Deca: $10 = 10^{1}$
* Hecta: $100 = 10^{2}$
* Kilo: $1,000 = 10^{3}$

	English to Metric	Metric to English
Weight/Mass	1 oz = 28.3495 g 1 lb = 0.4536 kg 1 t = 0.9072 Mg	1 g = 0.0353 oz 1 kg = 2.2046 lb 1 Mg = 1.1023 t
Volume	1 fl oz = 29.5735 mL 1 qt = 0.9464 L 1 gal = 3.7854 L	1 L = 1.0567 qt 1 L = 0.2642 gal 1 mL = 0.0338 fl oz
Distance	1 in = 2.54 cm 1 ft = 30.48 cm 1 yd = 0.9144 m 1 mi = 1.6093 km	1 cm = 0.3937 in 1 cm = 0.0328 ft 1 m = 1.0936 yd 1 km = 0.6214 mi

Index

ALPHABETICAL LIST OF CONCEPTS WITH PROBLEM NUMBERS

This comprehensive index organizes the concepts and skills discussed within the book alphabetically. Each entry is accompanied by one or more problem numbers, in which the topics are most prominently featured.

All of these numbers refer to problems, not pages, in the book. For example, 4.9 is the ninth problem in Chapter 4.

A

B

C

D

E–F

G–H

I–J–K

L–M

N–O

P

Q–R

S

T–U

V–W

X–Y–Z

The only way to learn basic math and pre-algebra is to do basic math and pre-algebra problems.

A LOT of them.

Sorry!

And that's what you get here—more basic math and pre-algebra problems than your worst nightmare—but with a BIG difference. I've been through the whole book and made a ton of notes, so you get ...

✓ More practice problems than you can shake a stick at

✓ Extra explanations that make what's baffling perfectly clear

✓ Pointers to other problems that show related skills

Even though LCMs of larger denominators (like in Problems 8.12–8.13) are usually easier to build using factor trees and prime factorizations, don't forget to try the multiples technique first, just in case.

With over 800 problems, all of the major players are here: basic binary operations, inequalities, factors, fractions, decimals, percentages, exponents, roots and radicals, variables, linear equations, basic geometry, weights and measures, basic statistics, and more. So dig in to your heart's content.

W. MICHAEL KELLEY has written a bunch of books about math, like The Humongous Book of Calculus Problems; The Humongous Book of Algebra Problems; The Humongous Book of Geometry Problems; The Humongous Book of Statistics Problems; The Complete Idiot's Guide® to Calculus, Second Edition; The Complete Idiot's Guide® to Precalculus; and The Complete Idiot's Guide® to Algebra, Second Edition.

ALPHA

Cover design © Alpha Books
penguin.com

U.S. $19.95
CAN $23.00

ISBN 978-1-61564-083-6

51995

9 781615 640836